

18.311 — MIT (Spring 2011)

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Problem Set # 04.

Due: Monday April 11.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages**, each with your **FULL NAME** clearly spelled.

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1 Regular Problems.

1.1 Statement: problem ExPD40. Normal modes for the heat equation with mixed boundary conditions.

Consider the problem

$$T_t = T_{xx} \quad \text{for } 0 < x < \pi, \quad (1.1)$$

with boundary conditions¹ $T(0, t) = 0$ and $T_x(\pi, t) = 0$. Then

1. **Find all the normal mode solutions** — that is: solutions of the form $T = e^{\lambda t} \phi(x)$.
2. **Show that** the associated eigenvalue problem² is self-adjoint. Hence the spectrum is real, and the eigenfunctions can be used to represent “arbitrary” functions. In this case the spectrum is a discrete set of eigenvalues $\{\lambda_n\}$, and the eigenfunctions form an orthogonal basis.
3. Given initial conditions $T(x, 0) = T_0(x)$, **write the solution to (1.1) as a series** involving the normal modes

$$T = \sum_n a_n e^{\lambda_n t} \phi_n(x). \quad (1.2)$$

Write explicit expressions³ for the coefficients a_n .

1.2 Statement: problem ExPD41. Normal modes for the heat equation with Robin boundary conditions.

Consider the problem

$$T_t = T_{xx} \quad \text{for } 0 < x < \pi, \quad (1.3)$$

with boundary conditions⁴ $T(0, t) = 0$ and $T_x(\pi, t) = T(\pi, t)$. Then

1. **Find all the normal mode solutions** — that is: solutions of the form $T = e^{\lambda t} \phi(x)$.

¹Fixed temperature on the left and no heat flux on the right.

²That is: $\lambda \phi = \phi_{xx}$, with the appropriate boundary conditions.

³They should be given by integrals involving T_0 and the eigenfunctions ϕ_n .

⁴Fixed temperature on the left. The right boundary condition is known as a Robin condition. It arises in situations where the heat flux at a boundary is proportional to the temperature difference with some cooling fluid flowing by.

2. **Show that** the associated eigenvalue problem⁵ is self-adjoint. Hence the spectrum is real, and the eigenfunctions can be used to represent “arbitrary” functions. In this case the spectrum is a discrete set of eigenvalues $\{\lambda_n\}$, and the eigenfunctions form an orthogonal basis.
3. Given initial conditions $T(x, 0) = T_0(x)$, **write the solution to (1.3) as a series** involving the normal modes

$$T = \sum_n a_n e^{\lambda_n t} \phi_n(x). \quad (1.4)$$

Write explicit expressions⁶ for the coefficients a_n .

1.3 Statement: Longitudinal vibrations of an elastic rod.

This problem consists of **5 questions that you must answer** (see below).

Consider a slender cylindrical rod of same elastic material, lined up with the x axis — i.e.: the axis of the cylinder lies along the x coordinate axis. Assume now that all the motion occurs along the x axis.⁷ In particular, assume that there is neither bending nor torsion of the rod. In this situation, *we can describe the rod, at any any time, by a single function*

$$X = X(x, t) = \begin{cases} \text{Position on the } x \text{ axis, at time } t, \text{ of the point in the rod} \\ \text{whose position is } x \text{ when the rod is at equilibrium.} \end{cases} \quad (1.5)$$

Here by equilibrium we mean that there are no internal (elastic) forces in the rod. Hence we can write $\mathbf{X} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$, where \mathbf{u} denotes the displacement of any point in the rod from its equilibrium position. When the rod is not at equilibrium, forces occur. In the case of an elastic body, these forces depend on how stretched or compressed the rod is at each point. Before we can write an expression for the elastic forces, we **need a measure of the rod stretching**:

Consider what happens with an infinitesimal length dx , under the transformation $x \rightarrow X$. Clearly: $dx = (x + dx) - x \longrightarrow X(x + dx, t) - X(x, t) = (1 + u_x) dx$. Thus, at every point, u_x measures how stretched (or compressed, for $u_x < 0$) the material of the rod is, with $1 + u_x$ the ratio of the

⁵That is: $\lambda \phi = \phi_{xx}$, with the appropriate boundary conditions.

⁶They should be given by integrals involving T_0 and the eigenfunctions ϕ_n .

⁷This is an *approximation*. If a rod is stretched, it does (generally) become thinner. Thus motion perpendicular to the rod axis occurs. However, if the stretching is not too large, it is reasonable to neglect the transversal motion.

current length relative to the equilibrium length. *The (elastic) forces on the rod are then as follows:*

$$\left. \begin{array}{l} \text{At each point on the rod (say, the one that at equilibrium is at } x = x_*), \text{ the portion} \\ \text{of the rod ahead (i.e.: } x > x_*) \text{ applies a force } \mathbf{F} \text{ on the portion behind (i.e.: } x < x_*). \\ \text{Viceversa, } x < x_* \text{ applies an opposite and equal force } -\mathbf{F} \text{ on } x > x_*. \end{array} \right\} \quad (1.6)$$

QUESTION 1. Guive a physical argument for why the two forces must be equal and opposite.

Hint: what would happen with the infinitesimal amount of mass at x_* if they were not?

Because the rod is elastic, F is a function of u_x , namely
$$F = F(u_x), \quad (1.7)$$

where we assume that the rod is homogeneous, so that the

relationship between force and deformation is the same all along the rod.

Note 1. If the rod is stretched ($u_x > 0$) each side should pull on the other ($F > 0$). Viceversa, a push ($F < 0$) should result from compression ($u_x < 0$). In general, we expect that:

$$F(0) = 0, \quad \text{and} \quad \frac{dF}{du_x} > 0. \quad (1.8)$$

In particular, for **small deformations** (u_x small), we can approximate F by the first (significant) term in its Taylor expansion (assuming that F is smooth), and write

$$F \cong k u_x, \quad \text{where } k > 0 \text{ is a constant.} \quad (1.9)$$

This is **Hooke's law of elasticity**.

Note 2. A uniform displacement of a rod at elastic equilibrium should keep the rod at equilibrium. This is borne out by the equations above, since a uniform displacement corresponds to $u = u(t)$. There are very many possible positions where a rod is at elastic equilibrium. In defining X above, we must select one of them. Of course, it is convenient⁸ to select one in an inertial frame of reference, so that no 'inertial' forces appear when writing the equations for the rod behavior. Assume that this is the case here.

QUESTION 2. Using the methods introduced in class, derive an equation (a p.d.e.) for the function $u = u(x, t)$, assuming that the only forces affecting the rod behavior are the elastic forces described above. Assume that the mass per unit length of the rod at equilibrium is a constant, ρ .

⁸Though not required!

Hints. Note that the velocity of any material point on the rod, characterized by its equilibrium position x , is given by $u_t(x, t)$. Thus the momentum in any piece of the rod, $a < x < b$, is given by $\int_a^b \rho u_t dx$. Write now an equation for the conservation of the momentum in $a < x < b$, then use the fact that $a < b$ are arbitrary to derive a p.d.e. for u .

QUESTION 3. What form does the p.d.e. you just derived have when (1.9) applies?

QUESTION 4. What dimensions does the constant $c = \sqrt{k/\rho}$ have?

QUESTION 5. In general, what dimensions does $c = c(u_x) = \sqrt{\frac{1}{\rho} \frac{dF}{du_x}}$ have? Notice that u_x is dimension-less.

Remark. An elastic rod can exhibit other types of motion, in addition to the longitudinal one described here. In particular: bending, and torsion (we may come back to these later in the course). Each of these is associated with a particular type of waves that can propagate along a rod. For example: P-waves (longitudinal vibrations), and S-waves (transverse vibrations).

2 Special Problems.

2.1 Statement: problem ExPD20.

Separation of variables for the 1D wave equation.

Consider the wave equation

$$u_{tt} - u_{xx} = 0. \tag{2.10}$$

Elsewhere it was shown that any solution of this equation can be written in the form:

$$u = f(x - t) + g(x + t), \tag{2.11}$$

where f and g are arbitrary functions. **Find ALL the separation of variables solutions to the wave equation (2.10), for the cases below** — the separation of variables solutions are those that have the form $u = T(t) X(x)$, for some functions $T = T(t)$ and $X = X(x)$.

- **Case A.** The solution $u = u(x, t) = T(t) X(x)$ is periodic in x , of period 2π .
- **Case B.** The solution $u = u(x, t) = T(t) X(x)$ vanishes at $x = 0$, and at $x = \pi$.

The second case corresponds to the problem of a string of (nondimensional) length π , tied at both ends. The solutions that you will find are the *standing wave* modes for the wave equation.

In each case, show that the solutions you have found have the form given in equation (2.11) above. In particular, this will show you how the standing wave solutions can be written as a combination of a left and a right moving waves.

2.2 Statement: Solitary wave for the KdV equation.

2.2.1 Introduction.

By introducing a diffusive term of the form $\nu \rho_{xx}$ (where ν is small and positive) into the equation for traffic flow, one can resolve the structure of shocks as **traveling waves**. That is, the equation

$$\rho_t + c(\rho) \rho_x = \nu \rho_{xx}, \quad \text{where } \frac{dc}{d\rho} \neq 0, \quad (2.12)$$

has smooth traveling wave solutions, that become discontinuous shock transitions as $\nu \downarrow 0$. To be precise, equation (2.12) has solutions of the form

$$\rho = f\left(\frac{x - Vt - x_0}{\nu}\right), \quad (2.13)$$

where V and x_0 are constants, and $f = f(\zeta)$ is a smooth function with the properties:

- A.** $f(\zeta) \rightarrow \rho_0$ as $\zeta \rightarrow -\infty$.
- B.** $f(\zeta) \rightarrow \rho_1$ as $\zeta \rightarrow \infty$.

Thus, as $\nu \downarrow 0$, the solution above in (2.13) becomes a discontinuity (shock), traveling along the line $x = x_0 + Vt$, and connecting the state ρ_0 behind with the state ρ_1 ahead. For $\nu > 0$ small, but finite, the discontinuity is resolved by this solution into a smooth transition (over a length scale proportional to ν) connecting the two sides of the shock jump. Furthermore: the **Entropy** and **Rankine Hugoniot** jump conditions also *follow from these solutions*, since a function f with the properties above exists if and only if

$$c(\rho_0) > c(\rho_1) \quad \text{and} \quad V = \frac{[q]}{[\rho]}. \quad (2.14)$$

In the particular case when c is a linear function of ρ (quadratic flow function $q = q(\rho)$), it is easy to see that equation (2.12) reduces to the **Burgers equation** for the characteristic speed c . Namely:

$$c_t + cc_x = \nu c_{xx}. \quad (2.15)$$

This follows upon multiplying (2.12) by $\frac{dc}{d\rho}$ (a constant in this case) and using the chain rule.

The Burgers equation has smooth traveling waves $c = f\left(\frac{x - Vt - x_0}{\nu}\right)$, that can be written explicitly in terms of elementary functions:

$$c = \frac{c_0 + c_1}{2} - \frac{c_0 - c_1}{2} \tanh\left(\frac{c_0 - c_1}{4} \zeta\right), \quad \text{where } \zeta = \frac{x - Vt - x_0}{\nu} \text{ and } c_0 > c_1. \quad (2.16)$$

These connect the states c_0 as $x \rightarrow -\infty$ with c_1 as $x \rightarrow \infty$. The speed V is given by the appropriate shock relation $V = \frac{1}{2}(c_0 + c_1)$, and x_0 is arbitrary.

There are many conservative processes in nature where (at leading order) a nonlinear kinematic first order equation applies (i.e.: the same equation as in traffic flow, with some flow function $q = q(\rho)$, that depends on the details of the processes involved). In all these cases the leading order equations lead to wave steepening and breaking, that generally is stopped by the presence of physical processes that become important only when the gradients become large. However, **it is not generally true** that these higher order effects are dominated by dissipation — many other possibilities can occur. Below we consider one such alternative situation, which happens to be quite common

Remark 2.1 *The point made in the prior paragraph is very important, since shocks (as a resolution of the wave breaking caused by the kinematic wave steepening) are the answer **only** when the higher order effects are of a dissipative nature. One should be very careful about not introducing shocks into mathematical models of physical processes just because the models exhibit wave distortion and breaking. Many other behaviors are possible, some rather poorly understood. The answer to a mathematical breakdown in a model is almost never to be found purely by mathematical arguments; a careful look at the physical processes the equations attempt to model is a must when this happens.*

A possible, and quite frequent, alternative to dissipation is dispersion: namely, the wave speed is a non-trivial function of the wave number. The simplest instance of dispersion introduces a term proportional to the third space derivative of the solution in the equations. When coupled with the simplest kind of nonlinearity (quadratic), this gives rise to the **Korteweg-de Vries (KdV) equation**. The nondimensional form of the KdV equation is:

$$\mathbf{u}_t + \mathbf{u} \mathbf{u}_x = \epsilon^2 \mathbf{u}_{xxx}, \quad (2.17)$$

where $\epsilon > 0$ is a parameter expressing the ratio of dispersion (different wavelengths moving at different speeds) to nonlinearity.

Remark 2.2 *The KdV equation describes (for example) the propagation of long, small (but finite) amplitude, waves in shallow water channels. The KdV equation is a “canonical” equation that arises in very many dispersive situations. That this should be so is relatively simple to see:*

First. *Consider a wave situation where the wave phase speed depends on the wave number in a nontrivial way: $c_p = c_p(k)$. Clearly, c_p should be an even function of k , since both k and $-k$ correspond to the same wavelength. Thus, for long waves (k small) one should be able to expand c_p in the form $c_p = \alpha + \beta k^2 + \dots$ (where α and β are constants). Furthermore, by changing coordinates into a moving frame, we can always assume $\alpha = 0$.*

Second. *$c_p \approx \beta k^2$ corresponds to a relationship between the wave number and the wave frequency of the form $\omega = k c_p(k) \approx \beta k^3$, which corresponds to the equation $u_t = \beta u_{xxx}$. Thus we see how a third order derivative arises as the simplest example of dispersive behavior for long waves.*

2.2.2 The problem to do.

Show that: **for the KdV equation above in (2.17), no shocks are possible.** To be precise, consider the non-trivial (i.e. $u \neq \text{constant}$) traveling wave solutions of (2.17):

$$u = F(\zeta), \quad \text{where} \quad \zeta = \frac{x - Vt - x_0}{\epsilon}, \quad (2.18)$$

$F = F(\zeta)$ is some smooth function, V is a constant (the wave speed), and x_0 is some arbitrary constant. Study all the solutions of this form such that (for some constants u_0 and u_1)

(a) $F(\zeta) \rightarrow u_0$ as $\zeta \rightarrow -\infty$, $F(\zeta) \rightarrow u_1$ as $\zeta \rightarrow \infty$, and F is not identically constant.

(b) The first and all the higher order derivatives of $F = F(\zeta)$ vanish as $\zeta \rightarrow \pm\infty$.

Then

$$\boxed{\text{Show that the conditions: } V < u_0 = u_1, \text{ must apply.}} \quad (2.19)$$

These solutions cannot represent shock transitions, since the states at $\pm\infty$ are equal (therefore, no jump occurs). In fact, it is possible to write these solutions explicitly in terms of elementary functions (hyperbolic secant), but finding this explicit form is **optional**.

Remark 2.3 *The traveling waves described above (that you are supposed to analyze) are called **solitary waves**, because they consist of a single isolated disturbance that vanishes very quickly as $x \rightarrow \pm\infty$. Such waves are easy to see in shallow water situations, where disturbances of a wavelength much bigger than the depth are generated. For example, in early summer many lakes develop a thin layer (a few feet thick) of warm water over the colder (heavier) rest of the water. Waves on this top layer, with wavelengths much longer than its thickness, obey an equation similar to the KdV equation, also supporting solitary waves. These are easy to see (when they are generated by boats) as traveling bumps on the surface of the lake — these are “one-dimensional” bumps, that is: they decay only in the direction of propagation; the surface elevation is (basically) independent of the direction normal to the propagation direction (thus they look like “rolls”, moving on the lake surface).*

Note that the solitary waves for (2.17) are actually “dips”, not bumps. In order to obtain “bumps”, the sign of the dispersive term in (2.17) must be reversed. Namely, one must consider the equation

$$v_t + v v_x = -\epsilon^2 v_{xxx}.$$

The traveling waves for this equation, and those for (2.17), are related. In fact: $u = f(x - V t)$ is a traveling wave for (2.17), if and only if $v = -f(x + V t)$ is a traveling wave for the equation above.

Hint 2.1 *First: substitute (2.18) into the KdV equation (2.17). This will yield a third order O.D.E. for the function $F = F(\zeta)$. Notice that the parameter ϵ is used in (2.18) to scale the wavelength of the traveling wave, precisely in the form needed to eliminate ϵ from the O.D.E. that F satisfies.*

Second: you should be able to integrate the third order O.D.E just obtained once, and reduce it to a second order O.D.E. — with some arbitrary integration constant. If you multiply this second order O.D.E. by $\frac{dF}{d\zeta}$, the result can again be integrated, so that you will end up with a first order O.D.E. for F — with two integration constants in it.

What does “to integrate an O.D.E.” mean?

To integrate an O.D.E., you must first write it in the form derivative (something) = 0, from which you can conclude that something = constant — you have just “integrated” the O.D.E. once. Sometimes an O.D.E. cannot be written directly as the derivative of something, but it is still possible to do so by multiplying the equation by an appropriate integrating factor. This is what the multiplication by $dF/d\zeta$ in this hint will accomplish.

Finally, consider the limits as $\zeta \rightarrow \pm\infty$ of the O.D.E. you obtained in the previous steps. Using the assumed properties for F , you should now be able to obtain that $u_0 = u_1$. Showing that you also need $V < u_0 = u_1$ for a solution to exist is a bit trickier, and will require you to do some analysis of the equation — similar to the one used to study the traveling wave solutions of (2.12).

2.2.3 First order O.D.E. review.

Here we review some facts about real valued solutions to O.D.E.'s of the form

$$\left(\frac{dy}{dx}\right)^2 = R(y), \quad (2.20)$$

where R is a real valued, smooth function. We consider the case $R = (y - a)(y - b)^2$ only, where a and b are constants. However, the same type of analysis can be used for R of general form, by considering the zeros of R — i.e.: the points y_* at which $R(y_*) = 0$. In particular, note that the case $R = c(y - a)(y - b)^2$, where $c > 0$ is a constant, can be reduced to this one by the simple scaling $x \rightarrow \sqrt{c}x$. We will also **assume that $a \neq b$** , since $a = b \implies y = a + \frac{4\delta}{(x - x_0)^2}$ (where x_0 is a constant and $\delta = 0$ or $\delta = 1$) and there is not much to analyze.

The **process below is based in the following idea, useful for any R** : once we know the behavior of the solutions of (2.20) near the zeros of R , and for $|y|$ large, the overall qualitative behavior is easy to ascertain. This is because the sign of $\frac{dy}{dx}$ can change only at the zeros of R , so that $y = y(x)$ must be monotone between zeros.

Imagine the solutions plotted in the (x, y) -plane — where we draw the horizontal lines $y = y_*$, for every zero of R . Now consider a solution in any horizontal strip between two such zeros: it will have to be monotone (increasing or decreasing) till it reaches one of the zeros. The **key question is: can a zero be reached at a finite value of x** ? If not, then the solution will reach the zero **only** as either $x \rightarrow \infty$, or $x \rightarrow -\infty$. If yes, then we need to know what happens when the zero is reached, to continue the solution beyond it.

It turns out that only first order zeros can be reached at a finite value of x , and there the solution “bounces” back from the zero, turning from monotone increasing to monotone decreasing (or vice versa), with a sign change in $\frac{dy}{dx}$. Thus first order zeros of R are associated with local maximums (or minimums) of the solutions.

We must also study the solutions in the half planes between the largest (smallest) zero of R and ∞ (resp. $-\infty$). For this we need to know the solutions' behavior for $|y|$ large.

FIRST: It is clear that we **need** $y \geq a$. Else the right hand side in (2.20) is negative and the solution cannot be real valued — **except, possibly, for the trivial solution $y \equiv b$ (if $b < a$.)**

SECOND: Consider the behavior of the non-trivial solutions near the zeros of R (i.e.: exclude the solutions $y \equiv a$ and $y \equiv b$.)

- **For $y \approx a$,** write $y = a + z(x)$, with $z > 0$ small. Then $(dz/dx)^2 = (a - b)^2 z$ approximates the equation. This has the solutions $z = \frac{1}{4}(a - b)^2(x - x_0)^2$, where x_0 is a constant. Thus it is clear that the solutions of (2.20) will reach local minimums when y approaches a .
- **For $y \approx b$,** write $y = b + z(x)$, with z small. Then $(dz/dx)^2 = (b - a) z^2$ approximates the equation. This has real solutions only if $b > a$, with $z = c e^{\kappa x}$, c a constant and $\kappa = \pm\sqrt{b - a}$. Thus the solutions of (2.20) can approach b in the limits $x \rightarrow \pm\infty$, but only if $b > a$.

THIRD: Use these results to analyze the real valued solutions of (2.20) in the two possible cases.

Case $a > b$. The solutions are real only if $y \geq a$ (except for the trivial solution $y \equiv b$). Then either $y \equiv a$ or $y > a$ somewhere. In this second case the solution has a minimum at some $x = x_0$ with $(dy/dx) > 0$ for $x > x_0$ and $(dy/dx) < 0$ for $x < x_0$. Away from x_0 , the solution grows without bound. Eventually y becomes very large, $(dy/dx)^2 \approx y^3$, and the solutions blow up like $4/(x - x_*)^2$.

Thus the only bounded real solutions are the trivial ones $y \equiv a$ and $y \equiv b$.

Case $b > a$. If $y > b$ anywhere, the solution decays to $y = b$ as either $x \rightarrow \pm\infty$, with a singularity like $4/(x - x_*)^2$ at a finite $x = x_*$. This follows because either $(dy/dx) > 0$ or $(dy/dx) < 0$ and there is no way for the sign to change. On the other hand, if $a < y < b$, the solution decreases from $y = b$ at $x = -\infty$, to a minimum at some $x = x_0$ (where $y = a$) and then increases back to $y = b$ at $x = \infty$.

In this case nontrivial bounded solutions exist in the range $a \leq y < b$. These have limits $y = b$ at $x = \pm\infty$ and a single minimum (where $y = a$) at some finite $x = x_0$.

The analysis above depends only on the nature of the zeros of R . For $R = (y - a)(y - b)^2$ we can **solve the equation explicitly** and verify the results: Introduce $z = z(x)$ by $z^2 = y - a$ and write $\nu^2 = b - a$. Equation (2.20) — i.e.: $(y')^2 = (y - a)(y - b)^2$ — then becomes

$$4z^2(z')^2 = z^2(z^2 - \nu^2)^2 \iff 2z' = \pm(z^2 - \nu^2) \iff \ln\left(\frac{z - \nu}{z + \nu}\right) = \pm\nu x + \alpha,$$

where the prime denotes differentiation with respect to x and α is an arbitrary constant. Here we do not make any assumptions on the signs of $y - a$ and $b - a$: **thus z , ν and α need not be real.** With some further manipulation this yields, in terms of λ (defined by $\lambda^2 = -\exp(\pm\nu x + \alpha)$),

$$z = \nu \frac{1 - \lambda^2}{1 + \lambda^2} \iff \nu^2 - z^2 = \nu^2 \left(\frac{2}{\lambda + \lambda^{-1}} \right)^2 \iff y = b - \nu^2 \left(\frac{2}{\lambda + \lambda^{-1}} \right)^2 .$$

Finally we can write, being now careful with keeping things real:

Case $a > b$. Let $\mu > 0$ be defined by $\mu^2 = a - b$. Then the nontrivial real solutions are

$$y = b + \mu^2 \sec^2 \left(\frac{1}{2} \mu (x - x_0) \right), \quad (2.21)$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$.) The relationship with the constants defined earlier is $\nu = \pm i\mu$ and $\alpha \pm \nu x_0 = i\pi$. See the left frame in figure 2.1.

Case $b > a$. Let $\mu > 0$ be defined by $\mu^2 = b - a$. Then the nontrivial real solutions are

$$y = b - \mu^2 \operatorname{sech}^2 \left(\frac{1}{2} \mu (x - x_0) \right) \quad \text{and} \quad y = b + \mu^2 \operatorname{cosech}^2 \left(\frac{1}{2} \mu (x - x_0) \right), \quad (2.22)$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$.) The relationship with the constants defined earlier is $\nu = \pm\mu$ and either $\alpha \pm \nu x_0 = i\pi$ or $\alpha \pm \nu x_0 = 0$. See the right frame in figure 2.1.

Next we briefly summarize the **situation for more general forms of R** — which can be analyzed with the same techniques we used here. For non-trivial real valued solutions:

1. At a first order zero of R , the solutions will either achieve a local minimum — if $dR/dy > 0$ there, or a local maximum — if $dR/dy < 0$ there. In other words, solutions can achieve values that correspond to simple zeros of R at some finite value of x , where they will have a local maximum or a local minimum. But these values cannot occur in the limits $x \rightarrow \pm\infty$.
2. By contrast, solutions cannot achieve values that correspond to higher (bigger than one) order zeros of R , at any finite value of x . Such values can be achieved only in the limits where $x \rightarrow \pm\infty$. For double zeros of R , the approach to the value (as $x \rightarrow \pm\infty$) is exponential. For zeros of order higher than two, the approach is algebraic .
3. Periodic oscillatory solutions occur in the regions between simple zeros, where $R > 0$. This situation does not occur in the example treated earlier, which has a single simple zero.

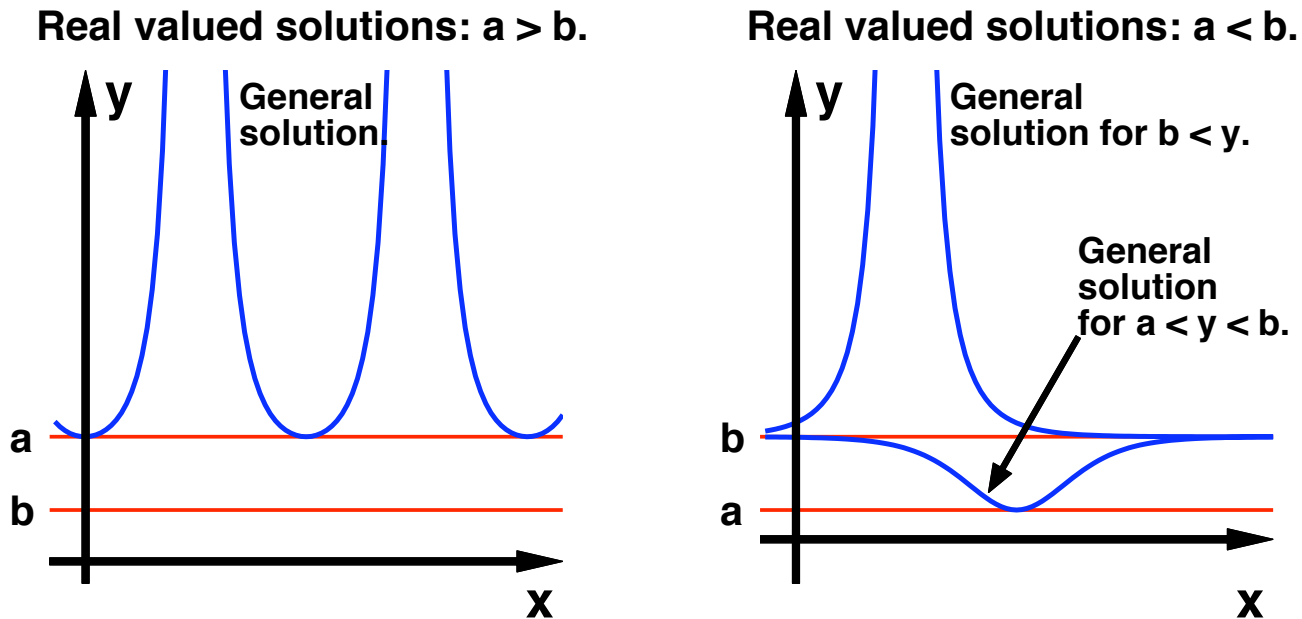


Figure 2.1: Real valued solutions of equation (2.20), with $R = (y - a)(y - b)^2$.

- Left frame: case $a > b$. The non-trivial solutions are periodic, with period $T = 2\pi/\sqrt{a - b}$.
- Right frame: case $a < b$. All the solutions satisfy $y \rightarrow b$ as $|x| \rightarrow \infty$.

4. Single bump (or dip) solutions occur in the regions, comprised between a simple and a double (or higher) order zero, where $R > 0$. The right frame in figure 2.1 shows an example of this.
5. The behavior of the solutions in regions where $R > 0$, comprised between a zero of R and $\pm\infty$, depends on the behavior of $R(y)$ for large values of y . If this leads to the formation of singularities (e.g.: $R = R(y)$ grows faster than linear as $y \rightarrow \pm\infty$), then the solutions are as in figure 2.1: either a periodic array of singularities (for a simple zero, left frame) or a single singularity — with decay to the value of y at the zero of R — as $x \rightarrow \pm\infty$ (for a double or higher order zero, right frame.)

THE END.