18.311 — MIT (Spring 2011)

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March 15, 2011.

Problem Set # 03.

Due: Monday March 28.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: TFPa22. Two space D and time characteristics.

If $\psi = \psi(x,\, y,\, t)$ solves the equation

$$\psi_t + u\,\psi_x + v\,\psi_y = 0\,, \tag{1.1}$$

show that ψ is constant on the lines

$$\frac{dx}{dt} = u, \quad \text{and} \quad \frac{dy}{dt} = v. \tag{1.2}$$

1.2 Statement: TFPa24.

Semi-linear first order equation and characteristics.

Consider the equation

$$x^2\,\psi_x + x\,y\,\psi_y = \psi^2\,, \tag{1.3}$$

subject to $\boldsymbol{\psi = 1}$ on the curve $\boldsymbol{\Gamma}$ given by $\boldsymbol{x = y^2}$. This is a semi-linear problem that can be written in terms of characteristics.

A. Compute the characteristic curves that cross the curve Γ, as follows: (i) Parameterize the curve Γ, say: $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{-\infty < \xi < \infty}$. (ii) Write the o.d.e.'s for the characteristic curves, in terms of some parameter (say, s) along each curve. (iii) Solve the o.d.e.'s for the characteristics, with the condition that $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{s = 0}$.

B. Describe what type of curves, in the x-y plane, are the characteristics. Which region of the plane do these curves cover? What happens with the characteristic corresponding to $\xi = 0$?

C. Solve the o.d.e. that ψ satisfies along each characteristic. Eliminate the parameters ξ and s in terms of x and y, and write an explicit formula for the solution $\psi = \psi(x,\,y)$ to (1.3).

D. Where is the solution ψ defined? **Hint:** *be careful with your answer here!*

2 Special Problems.

2.1 Statement: TFPb14. Check if discontinuities are allowed shocks.

Consider the conservation equation $\boldsymbol{\rho_t + q_x = 0}$ (for the conserved density[1] ρ) with various choices for the flow rate $q = q(\rho)$ — as given below. Assume that the underlying physical processes behind this conservation equation lead to the formation of shocks — as a resolution of the wave–breaking caused by the crossing of the characteristics. In each case the following *discontinuous solution is proposed*

$$\rho(x,t) = \begin{cases} \rho_l & \text{for} \quad x < 0 \quad \text{and all} \quad t > 0, \\ \rho_r & \text{for} \quad x > 0 \quad \text{and all} \quad t > 0, \end{cases} \tag{2.4}$$

[1] As far as this problem is concerned, what stuff ρ is the density for is not important. All you need to know is that the stuff is conserved. In fact, ρ need not even be the "absolute" density, it can just be the deviation from some reference density of stuff.

where ρ_l and ρ_l are constants.

In each case, check if the given proposed solution is actually a solution. You must justify your answers: If the proposed solution is actually a solution, verify this and, if it is not, say why not. Further: explain what you are doing, in words. A bunch of calculations, without an explanation of what they mean, or why you are doing them, is not an acceptable answer.

A. $q = \rho(2-\rho)$, with $\rho_l = 0.5$, $\rho_r = 1.5$, $q_l = q_r = 0.75$, $c_l = 1$, and $c_r = -1$.

B. $q = \rho(2-\rho)$, with $\rho_l = 0$, $\rho_r = 1$, $q_l = 0$, $q_r = 1$, $c_l = 2$, and $c_r = 0$.

C. $q = \rho^2$, with $\rho_l = -1$, $\rho_r = 1$, $q_l = q_r = 1$, $c_l = -2$, and $c_r = 2$.

D. $q = 1-\rho^2$, with $\rho_l = -1$, $\rho_r = 1$, $q_l = q_r = 0$, $c_l = 2$, and $c_r = -2$.

E. $q = \rho(2-\rho)$, with $\rho_l = 1.25$, $\rho_r = 0.75$, $q_l = q_r = 0.9375$, $c_l = -0.5$, and $c_r = 0.5$.

Remark 2.1 *The following* **notation is used above:** $q_l = q(\rho_l)$, $q_r = q(\rho_r)$, $c_l = c(\rho_l)$, *and* $c_r = c(\rho_r)$ *— where* $c = c(\rho) = \dfrac{dq}{d\rho}$ *is the characteristic speed.*
By the way: you can trust that whatever values I tell you q_l, q_r, c_l, *and* c_r *have (for the given flux* q *and densities* ρ_l *and* ρ_r*) is correct. This problem is definitely* **not** *about verifying arithmetic!*

THE END.