18.311 — MIT (Spring 2011)

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Problem Set # 02.

Due: Friday March 18.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 71.07.

Consider two moving observers (possibly far apart), both moving at the same velocity V, such that the number of cars the first observer passes is the same as the number passed by the second observer.

(a) Show that $V = \dfrac{\Delta q}{\Delta \rho}$.

(b) Show that the average density between the two observers stays a constant.

1.2 Statement: Haberman problem 73.02 and 03.

Assume that the car flow velocity is related to the car density by:

$$u = u_m \left(1 - \frac{\rho}{\rho_j}\right) \implies q = u_m \left(1 - \frac{\rho}{\rho_j}\right) \rho \implies c = u_m \left(1 - 2\,\frac{\rho}{\rho_j}\right), \tag{1.1}$$

where ρ_j is the jamming density and u_m is the car speed limit. Consider now the **red light turns green** problem and:

First: Calculate the maximum acceleration of a car which starts approximately one car length behind a traffic light (i.e. $x(0) = -1/\rho_j$).

Second: Calculate the velocity of a car at the moment it starts moving behind a light.

1.3 Statement: Haberman problem 74.01.

Assume that $u(\rho) = u_m\,(1 - \rho/\rho_j)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \ x < 0, \\ \rho_0\,(L - x)/L & \text{for } \ 0 \le x \le L, \\ 0 & \text{for } \ L < x, \end{cases} \tag{1.2}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, determine and sketch $\rho(x,\,t)$.

1.4 Statement: Haberman problem 77.03.

A **weak shock** is a shock in which the shock strength (the difference in densities) is small. For a weak shock, show that the shock velocity is *approximately* the average of the density wave velocities associated with the two densities. [**Hint:** Use Taylor series methods.]

1.5 Statement: Haberman problem 77.04..

Show $[\,\rho^2\,] \neq [\,\rho\,]^2$.

2 Special Problems.

2.1 Statement: Haberman problem 78.03.

Assume that $u = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$ and at $t = 0$ the traffic density is

$$\rho(x,\,0) = \begin{cases} (1/3)\,\rho_{\mathsf{max}} & \text{for } \ x < 0, \\ (2/3)\,\rho_{\mathsf{max}} & \text{for } \ x > 0. \end{cases} \tag{2.3}$$

Why does the density not change in time?

2.2 Statement: TFPa23. Traffic flow initial-boundary value problem.

Consider a semi–infinite ($\tilde{x} > 0$) highway for which the density at the entrance is

$$\tilde{\rho}(0,\,\tilde{t}) = \begin{cases} \tilde{\rho}_1 & \text{for } \ 0 < \tilde{t} < \tau, \\ \tilde{\rho}_0 & \text{for } \ \tau < \tilde{t}, \end{cases} \tag{2.4}$$

where $\tau > 0$ is constant, and such that the initial density is uniform along the highway: $\tilde{\rho}(\tilde{x},\,0) = \tilde{\rho}_0$, for $\tilde{x} > 0$. Furthermore, assume that $\tilde{\rho}_0$ is lighter traffic than $\tilde{\rho}_1$, and that both are light traffic; in fact assume that $\tilde{u}(\tilde{\rho}) = u_{\mathsf{max}}\,(1 - \tilde{\rho}/\rho_{\mathsf{max}})$ and that $0 < \tilde{\rho}_0 < \tilde{\rho}_1 < \rho_{\mathsf{max}}/2$.

Part 1. Show that a-dimensional variables can be found in which the situation takes the form

$$\rho_t + q_x = 0, \quad \text{with } \ \rho(x,\,0) = \rho_0 \ \text{ for } \ x > 0 \ \text{ and } \ \rho(0,\,t) = \begin{cases} \rho_1 & \text{for } \ 0 < t < 1, \\ \rho_0 & \text{for } \ 1 < t, \end{cases} \tag{2.5}$$

where $0 < \rho_0 < \rho_1 < 1/2$, $q = \rho\, u$, and $u(\rho) = (1 - \rho)$. In other words, find (dimensional) constants $x_*,\ t_*,\ \rho_*,\ u_*,\ \ldots$ such that $\tilde{x} = x_*\, x,\ \tilde{t} = t_*\, t,\ \tilde{\rho} = \rho_*\, \rho\ \ \tilde{u} = u_*\, u,\ \ldots$ reduces (2.4) to (2.5).

Part 2. Solve the problem in (2.5), and determine the density $\rho = \rho(x,\, t)$ for **all** $x > 0$ and $t > 0$.

In particular:

a) Solve and display the equations for all the characteristics.

b) Solve and display the equations for all shocks (explicitly).

c) Sketch the characteristics and shock paths in space-time.

d) Find explicit expressions for $\rho = \rho(x,\, t)$.

e) Sketch ρ as a function of x for selected values of time (illustrate **all** possible cases).

THE END.