18.311 — MIT (Spring 2011)

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Problem Set # 01.

Due: Tuesday March 08.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 57.06.

Consider an infinite number of cars, each designated by a number β. Assume that the car labeled by β starts from $x = \beta$ ($\beta > 0$) with zero velocity, and also assume it has a constant acceleration β.

(a) Determine the position and velocity of each car as a function of time.

(b) Sketch the path of a typical car.

(c) Determine the velocity field $u = u(x,\, t)$.

(d) Sketch the curves along which $u = u(x,\, t)$ is constant.

1.2 Statement: Haberman problem 58.02.

Assume that the probability P of exactly one car being located in any fixed short segment of highway of length Δx is approximately proportional to the length, $P = \lambda \Delta x$. Also assume that the probability of two or more cars in that segment of highway is negligible.[1]

(a) Show that the probability of there being exactly n vehicles along a highway of length x, $P_n(x)$, satisfies the **Poisson distribution,**

$$P_n(x) = \frac{1}{n!}\,(\lambda\, x)^n\, e^{-\lambda\, x}. \tag{1.1}$$

Hint: *Consider $P_n(x + \Delta x)$, and form a differential equation for $P_n(x)$, (see section 3.6).*

(b) Evaluate and interpret the following quantities

1. $P_n(0)$, $n \neq 0$.
2. $P_0(x)$, $x \neq 0$.
3. $P_1(x)$, $x \neq 0$.

(c) Calculate the expected number of cars on a highway of length x. Interpret your answer.

[1] Actually, you may assume that these approximations get better and better as $\Delta x \to 0$.

1.3 Statement: Haberman problems 61.01 & 61.02.

Many state laws say that, for each 10 m.p.h (16 k.p.h) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 feet (5 meters)). There is another law that gives a maximum speed limit (assume that this is 50 m.p.h (80 k.p.h)). **Find the flow of cars as a function of density, $q = q(\rho)$.**

The state laws on following distances stated in the prior paragraph, were developed in order to prescribe spacing between cars such that rear-end collisions could be avoided, as follows:

(a) Assume that the car immediately ahead stops instantaneously. How far would the car following at $\boldsymbol{u}$ m.p.h travel, if

(1) the driver's reaction time was $\boldsymbol{\tau}$, and

(2) after a $\boldsymbol{\tau}$ delay, the driver decelerated at constant maximum deceleration $\boldsymbol{\alpha}$?

(b) The calculation in part **(a)** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate $\boldsymbol{\alpha}$, but the driver following still takes time $\boldsymbol{\tau}$ to react. How far back does a car have to be, traveling at $\boldsymbol{u}$ m.p.h, in order to prevent a rear-end collision?

(c) Show that the law described in the first paragraph of this problem corresponds to part **(b)**, if the human reaction time is about 1 second and the length of a car is about 16 feet (5 meters).

Note: *What part* **c** *is asking you to do is to justify/derive the state law prescription, using the calculations in part* **b** *to arrive at a car-to-car separation that will avoid a collision when the cars are forced to brake.*

1.4 Statement: Haberman problem 72.01.

Show that if $u = u(\rho)$ is determined by braking distance theory (see exercise 61.2), then the waiting time per car after a traffic light turns green is the same as the human reaction time for braking.

1.5 Statement: Haberman problem 71.04.

Consider the p.d.e. $\rho_t + q_x = \rho_t + c(\rho)\,\rho_x = 0$, where $q = q(\rho)$ and $c = c(\rho) = \dfrac{dq}{d\rho}(\rho)$. Assume that $\rho = \rho(x,\,t)$ is a smooth solution of the equation, and let $c = c(x,\,t) = c(\rho)$. **Show that:**

$$c_t + c\,c_x = 0. \tag{1.2}$$

1.6 Statement: Haberman problem 73.01.

Assume that the traffic density is initially given by

$$\rho(x,\,0) = \begin{cases} \rho_{\mathsf{max}} & \text{for } \; x < 0, \\ \frac{1}{2}\,\rho_{\mathsf{max}} & \text{for } \; 0 < x < a, \\ 0 & \text{for } \; a < x, \end{cases} \tag{1.3}$$

where $a > 0$ is some fixed length, and that the car flow velocity is related to the car density by

$$u = u_{\mathsf{max}}\left(1 - \frac{\rho}{\rho_{\mathsf{max}}}\right) \implies q = u_{\mathsf{max}}\left(1 - \frac{\rho}{\rho_{\mathsf{max}}}\right)\rho \implies c = u_{\mathsf{max}}\left(1 - 2\,\frac{\rho}{\rho_{\mathsf{max}}}\right). \tag{1.4}$$

Sketch the initial density. Determine and sketch the density at all later times.

1.7 Statement: Haberman problem 74.02.

Assume that $u(\rho) = u_m\,(1 - \rho^2/\rho_j^2)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x < 0, \\ \rho_0\,(L - x)/L & \text{for } \; 0 < x < L, \\ 0 & \text{for } \; L < x, \end{cases} \tag{1.5}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, **determine and sketch** $\rho(x,\,t)$**.**

2 Special Problems.

2.1 Statement: Haberman problem 70.03.

Assuming nearly uniform, but heavy, traffic, show that in general it is impossible to prescribe the traffic flow at the entrance to a semi–infinite highway. In this situation, what might happen to cars waiting to enter the highway?

2.2 Statement: Haberman problem 74.03.

Consider the (non-dimensionalized) partial differential equation

$$\rho_t - \rho^2\,\rho_x = 0\,, \quad -\infty < x < \infty \ \text{ and } \ t > 0. \tag{2.6}$$

(a) Why can't this equation model a traffic flow problem?

(b) Solve this P.D.E. by the method of characteristics, subject to the initial conditions:

$$\rho(x,\,0) = \begin{cases} 1 & \text{for } \ x < 0, \\ 1 - x & \text{for } \ 0 \le x \le 1, \\ 0 & \text{for } \ 1 < x. \end{cases} \tag{2.7}$$

2.3 Statement: Dimensional analysis and diffusion speed.

In the lectures it was shown that if $\Theta = \Theta(\vec{x},\,t)$ denotes the concentration of salt in water (e.g.: grams per liter) then, **assuming that there is no motion** by the water

$$\Theta_t = \nu\,\Delta\,\Theta, \tag{2.8}$$

where ν is the diffusion coefficient — which we assume here to be constant[2] — and Δ is the *Laplacian operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The **same equation** (with a different value of ν) applies if, for example, Θ denotes sugar concentration, or the concentration of some coloring (e.g. ink).

Under the conditions where (2.8) applies, imagine that you inject a very tiny blob of ink inside the liquid.[3] Then the size of the blob will start increasing with time (due to the ink's diffusion). The blob's edge will cease to be sharp as time goes on, but here we will simplify things and assume that they remain sharp enough during the course of the experiment.

Make the approximation that, at time $t = 0$ (when you start the experiment) the blob is just a point. Then, using qualitative and physical arguments only **(no solution of the equation)**

1. Argue that the shape of the blob is a sphere for $t > 0$.
2. Find a formula for the radius of the blob $R = R(t)$ as a function of time. There is a numerical multiplicative constant α, as in $R = \alpha\,f(t)$, which you will not be able to determine (without solving the p.d.e.), but you should be able to get $f(t)$.

[2] Generally ν is not quite constant, since is depends (for example) on the temperature.

[3] Carefully, so as not to start motion. The ink density must match that of water, to avoid gravity induced motion.

Typical values for diffusion coefficients, as in the examples above, are $\boldsymbol{\nu = \gamma\, 10^{-5}\, \mathrm{cm}^2/\mathrm{sec}}$, where γ ranges[4] between 1/2 and 2. **Assume that $\boldsymbol{\alpha = \gamma = 1}$.** Then

3. What is the radius of the blob when $t = 1$ minute?

4. At what time does $R = 5$ cm?

These numbers should give you an idea of how long it would take to sweeten a cup of coffee if you just deposited a lump of sugar in it, and did not stir the coffee.

THE END.

[4] Larger molecules diffuse slower than smaller ones; thus $\gamma(\text{sugar}) < \gamma(\text{salt})$. Also, γ grows with temperature, and may even depend on the concentration.