18.311 — MIT (Spring 2010)

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Problem Set # 06.

Due: Tuesday April 06. Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Generic HINTS:

1. Always: check that your answers are sensible. *If it seems as if they predict something that contradicts physical observation, then something is probably wrong!*

Contents

1 Regular Problems.

1.1 Statement: problem ExPD02.
Initial value problem for the 1D wave equation.

In problem ExPD01 we showed that the general solution of the 1-D wave equation $u_{tt} - c^2 u_{xx} = 0$, with $c > 0$ a constant, has the form $u(x, t) = f(x - c\,t) + g(x + c\,t)$ — where f and g are arbitrary functions. Using this result, show that if initial values

$$u(x,\,0) = U(x) \quad \text{and} \quad u_t(x,\,0) = V(x), \tag{1.1}$$

are given for the wave equation, then the solution is

$$u = \frac{1}{2}\,(U(x-c\,t) + U(x+c\,t)) + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} V(s)\,ds. \tag{1.2}$$

In particular, show that **if $\boldsymbol{U}$ and $\boldsymbol{V}$ are periodic functions of $\boldsymbol{x}$ (of period $\boldsymbol{P}$) then:**

$\boldsymbol{u}$ is periodic in $\boldsymbol{x}$ (of period $\boldsymbol{P}$) and periodic in $\boldsymbol{t}$ (of period $\boldsymbol{T = P/c}$).

Use the GBNS_GScheme script in the 18311 MatLab Toolkit and check how closely the numerical calculation reproduces the periodicity in time (for initial conditions where, say, U is periodic of period 2 and $V \equiv 0$.) Note that in the GBNS_GScheme, $c = 1$.

1.2 Statement: problem ExPD20. Separation of variables for the 1D wave equation.

Consider the wave equation

$$u_{t\,t} - u_{x\,x} = 0. \tag{1.3}$$

Elsewhere it was shown that any solution of this equation can be written in the form:

$$u = f(x-t) + g(x+t), \tag{1.4}$$

where f and g are arbitrary functions. **Find ALL the separation of variables solutions to the wave equation (1.3), for the cases below** — the separation of variables solutions are those that have the form $u = T(t)\,X(x)$, for some functions $T = T(t)$ and $X = X(x)$.

- **Case A.** The solution $u = u(x,\,t) = T(t)\,X(x)$ is periodic in x, of period $2\,\pi$.
- **Case B.** The solution $u = u(x,\,t) = T(t)\,X(x)$ vanishes at $x = 0$, and at $x = \pi$.

The second case corresponds to the problem of a string of (nondimensional) length π, tied at both ends. The solutions that you will find are the *standing wave* modes for the wave equation.

In each case, show that the solutions you have found have the form given in equation (1.4) above. In particular, this will show you how the standing wave solutions can be written as a combination of a left and a right moving waves.

1.3 Statement: problem ExPD04.

Complex variables solution of the 2D Laplace equation.

By direct substitution, show that

$$u = f(x - iy) + g(x + iy), \tag{1.5}$$

(where f and g are "arbitrary" functions) is a solution of Laplace's equation:

$$u_{xx} + u_{yy} = 0. \tag{1.6}$$

2 Special Problems.

2.1 Statement: problem ExPD21.

Separation of variables: Laplace equation in a rectangle.

Consider the question of determining the steady state temperature in a thin rectangular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) plate are insulated. The solution to this problem can be written as the sum of the solutions to four problems of the form[1]

$$T_{xx} + T_{yy} = 0, \quad \text{for } 0 \leq x \leq \pi \quad \text{and } 0 \leq y \leq L, \tag{2.7}$$

with boundary conditions $\quad T(0,\, y) = T(\pi,\, y) = T(x,\, 0) = 0 \quad \text{and} \quad T(x,\, L) = h(x), \qquad (2.8)$
where $\boldsymbol{L > 0}$ is a constant, and $\boldsymbol{h = h(x)}$ is some prescribed function. In turn, *the solution to (2.7 – 2.8) can be written as a (possibly infinite) sum of separation of variables solutions.*

Separation of variables solutions to (2.7 – 2.8) are product solutions of the form

$$T(x,\, y) = \phi(x)\, \psi(y), \tag{2.9}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are some functions. In particular, from (2.8), it must be that

$$\phi(0) = \phi(\pi) = \psi(0) = 0 \quad \text{and} \quad h(x) = \phi(x)\, \psi(L). \tag{2.10}$$

[1] Here we use non-dimensional variables, selected so that one side of the plate has length π.

However, for the separated solutions, the function $\boldsymbol{h}$ **is NOT prescribed:** It is allowed to be whatever the form in (2.9) allows. The general h is obtained by adding separation of variables solutions.

These are the tasks in this exercise:

1. **Find ALL the separation of variables solutions.**
2. In exercise ExPD04 we stated that all the solutions to the Laplace equation — i.e.: (2.7) — must have the form $\boldsymbol{T = f(x - i\,y) + g(x + i\,y)}$, for some functions $\boldsymbol{f}$ and $\boldsymbol{g}$. *Note that these two functions must make sense for complex arguments, and have derivatives in terms of these complex arguments.* — which means that f and g must be analytic functions.

   Show that the result in the paragraph above applies to all the separated solutions found in item 1. Namely: find the functions f and g that correspond to each separated solution.

THE END.