18.311 — MIT (Spring 2010)

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Problem Set # 05.

Due: Tuesday March 30. Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Generic HINTS:

1. *When $q = q(\rho)$ is quadratic, the shock speed is the average of the characteristic speeds immediately to the right and left of the shock. For many problems this simplifies the algebra.*
2. *The general solution of a linear o.d.e. with a forcing term can be written as the sum of a particular solution, plus the general solution to the homogeneous problem.*
3. *When solving the o.d.e. for the shock path, as given by the Rankine-Hugoniot jump conditions, beware of the fact that many of the solutions in the problem sets are given by different formulas in different regions. Hence, as a shock enters a different region, the o.d.e. changes. Keep track of this. Example: a shock starts with constant velocity (states on each side are constant) and switches to variable velocity (enters a region with a rarefaction fan on one side).*
4. *Remember that the characteristics are the curves along which information propagates. A situation where the solution along a characteristic is determined backwards in time is* not physically meaningful, *as it violates causality.* Make sure that your solutions satisfy causality!
5. Always: check that your answers are sensible. *If it seems as if they predict something that contradicts physical observation, then something is probably wrong!*

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1 Special Problems.

1.1 Statement: Dispersive Waves and Modulations.

Consider the following linear partial differential equations for the scalar function $\boldsymbol{u = u(x,\, t)}$:

$$\begin{aligned} \boldsymbol{u_t + c\,u_x + d\,u_{xxx}} &= \boldsymbol{0}, && (1.1)\\ \boldsymbol{u_{tt} - u_{xx} + a\,u} &= \boldsymbol{0}, && (1.2)\\ \boldsymbol{i\,u_t + b\,u + g\,u_{xx}} &= \boldsymbol{0}, && (1.3) \end{aligned}$$

where the equations are written in a-dimensional variables, $\boldsymbol{(c,\, d,\, a,\, b,\, g)}$ are real constants, and $\boldsymbol{a > 0}$. These equations arise in many applications, but we will not be concerned with these here. It should be clear that, in all three cases,

$$u = Ae^{i\,(k\,x - \omega\,t + \theta_0)}, \quad \text{where} \quad \omega = \Omega(k), \tag{1.4}$$

is a solution of the equations, for any real constants $A > 0$, θ_0, k, and ω, provided that

M1. For equation (1.1): $\boldsymbol{\Omega(k) = c\,k - d\,k^3}$ ....**Verify that this is true.**

M2. For equation (1.2): $\boldsymbol{\Omega(k) = \pm\sqrt{a + k^2}}$ ....**Verify that this is true.**

M3. For equation (1.3): $\boldsymbol{\Omega(k) = -b + g\,k^2}$ ....**Verify that this is true.**

Remark 1.1 *Solutions such as that in (1.4) represent monochromatic sinusoidal traveling waves, with* **amplitude** $\boldsymbol{A}$, **phase** $\boldsymbol{\theta = k\,x - \omega\,t + \theta_0}$, **wave number** $\boldsymbol{k}$, *and* **angular frequency** $\boldsymbol{\omega}$. *The* **wave length** *and* **wave period** *are* $\boldsymbol{\lambda = 2\,\pi/k}$ *and* $\boldsymbol{\tau = 2\,\pi/\omega}$, *respectively. The wave profile's crests and troughs move at the speed given by* $\theta =$ *constant, namely: the* **phase speed** $\boldsymbol{c_p = \omega/k}$.

Remark 1.2 *In all three cases,* $\boldsymbol{\Omega = \Omega(k)}$ **is a real valued function of** $\boldsymbol{k}$**, with** $\boldsymbol{\dfrac{d^2\Omega}{dk^2}} \neq 0$ *— i.e.: Ω is not a linear function of k. Because of this, we say that the* **equations are dispersive** *and call* $\boldsymbol{\Omega}$ **the dispersion function.** *The (non-constant) velocity* $\boldsymbol{c_g = c_g(k) = d\omega/dk}$ *is called the* **group speed**, *and the objective of this problem is to find out what the meaning of c_g is.*

The reason for the name "dispersive" is as follows: In a dispersive system, waves with different wavelengths propagate at different speeds. Thus, a localized initial disturbance, made up of many modes of different wavelengths, will **disperse** *in time, as the waves cease to add up in the proper phases to guarantee a localized solution: Localization depends on destructive interference, outside some small region, of all the modes $a(k)\,e^{i\,(k\,x+\theta_0)}$ making up the initial disturbance. However, since these modes propagate at different speeds, the phase coherence needed for the destructive interference is destroyed by the time evolution.* **This phenomena is illustrated by figure** 1.1.

1.1.1 Tasks to be performed.

TASK 1. verify **M1** through **M3**, above below equation (1.4).

TASK 2: Consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number k and angular frequency ω are related by a dispersion relation

$$\omega = \Omega(k). \tag{1.5}$$

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a *local wave number*

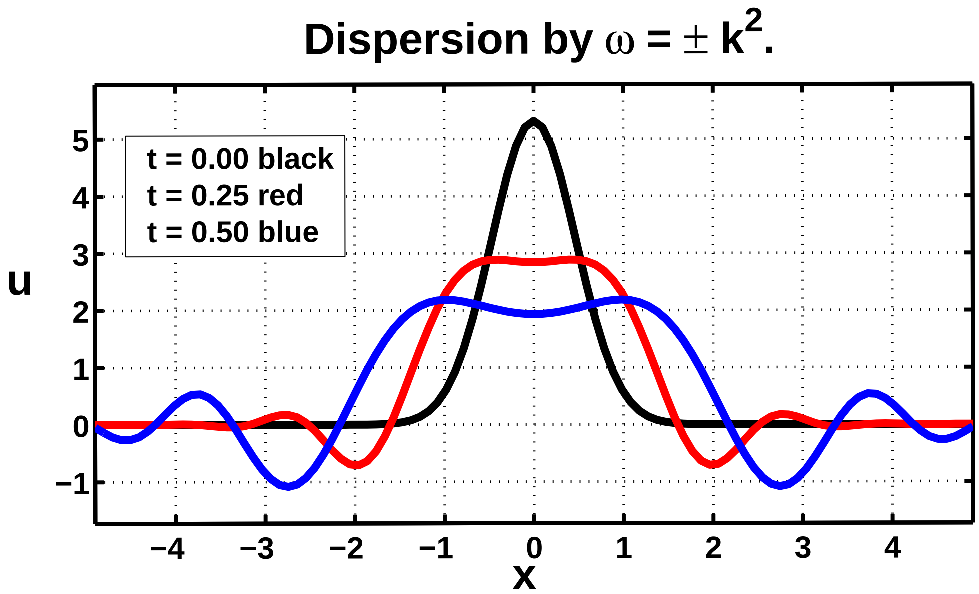

Figure 1.1: Example of dispersion: initial "Gaussian" bump, as it evolves under a dispersive equation with $\Omega(k) = \pm k^2$ — i.e.: $u_{tt} + u_{xxxx} = 0$. Solution at times $t = 0, 1/4, 1/2$ displayed. As the initial lump's phase coherence is destroyed by dispersion, localization is lost, and the bump "disperses".

$k = k(x,\, t)$ and a *local angular frequency* $\omega = \omega(x,\, t)$. In particular, assume that both k and ω vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods — *on the other hand, they may change considerably over many wavelengths or wave periods.* Then

Assuming conservation of wave crests, derive equations governing k and ω.

These equations are called the **Wave Modulation Equations.**

Remark 1.3 *Notice that the assumption that k and ω vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.*

Remark 1.4 **Why is it reasonable to assume that the wave crests are conserved?** *The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where* *the assumption of slow variation is invalid. There are some pretty interesting research problems in pattern formation that are related to this point.*

Hint 1.1 *It should be clear that one of the equations is $\omega = \Omega(k)$, since the solution behaves locally like a monochromatic wave. For the second equation, express the density of wave crests (and its flux) in terms of k and ω — in a sinusoidal wave:* How many wave crests are there per unit length? How many wave crests pass through a fixed point in space per unit time? *Then write the equation for the conservation of wave crests using these quantities.*

2 Regular Problems.

2.1 Statement: Solitary wave for the KdV equation.

2.1.1 Introduction.

By introducing a diffusive term of the form $\nu\rho_{xx}$ (where ν is small and positive) into the equation for traffic flow, one can resolve the structure of shocks as **traveling waves.** That is, the equation

$$\rho_t + c(\rho)\,\rho_x = \nu\,\rho_{xx}, \quad \text{where } \frac{dc}{d\rho} \neq 0, \tag{2.6}$$

has smooth traveling wave solutions, that become discontinuous shock transitions as $\nu \downarrow 0$. To be precise, equation (2.6) has solutions of the form

$$\rho = f\left(\frac{x - V\,t - x_0}{\nu}\right), \tag{2.7}$$

where V and x_0 are constants, and $f = f(\zeta)$ is a smooth function with the properties:

A. $f(\zeta) \to \rho_0$ as $\zeta \to -\infty$.

B. $f(\zeta) \to \rho_1$ as $\zeta \to \infty$.

Thus, as $\nu \downarrow 0$, the solution above in (2.7) becomes a discontinuity (shock), traveling along the line $x = x_0 + V\,t$, and connecting the state ρ_0 behind with the state ρ_1 ahead. For $\nu > 0$ small, but finite, the discontinuity is resolved by this solution into a smooth transition (over a length scale proportional to ν) connecting the two sides of the shock jump. Furthermore: the **Entropy** and

Rankine Hugoniot jump conditions also *follow from these solutions,* since a function f with the properties above exists if and only if

$$c(\rho_0) > c(\rho_1) \quad \text{and} \quad V = \frac{[\,q\,]}{[\,\rho\,]}. \tag{2.8}$$

In the particular case when c is a linear function of ρ (quadratic flow function $q = q(\rho)$), it is easy to see that equation (2.6) reduces to the **Burgers equation** for the characteristic speed c. Namely:

$$c_t + c c_x = \nu\, c_{xx}. \tag{2.9}$$

This follows upon multiplying (2.6) by $\dfrac{dc}{d\rho}$ (a constant in this case) and using the chain rule.

The Burgers equation has smooth traveling waves $c = f\left(\dfrac{x - V\,t - x_0}{\nu}\right)$, that can be written explicitly in terms of elementary functions:

$$c = \frac{c_0 + c_1}{2} - \frac{c_0 - c_1}{2}\,\tanh\left(\frac{c_0 - c_1}{4}\,\zeta\right), \quad \text{where } \zeta = \frac{x - V\,t - x_0}{\nu} \text{ and } c_0 > c_1. \tag{2.10}$$

These connect the states c_0 as $x \to -\infty$ with c_1 as $x \to \infty$. The speed V is given by the appropriate shock relation $V = \dfrac{1}{2}(c_0 + c_1)$, and x_0 is arbitrary.

There are many conservative processes in nature where (at leading order) a nonlinear kinematic first order equation applies (i.e.: the same equation as in traffic flow, with some flow function $q = q(\rho)$, that depends on the details of the processes involved). In all these cases the leading order equations lead to wave steepening and breaking, that generally is stopped by the presence of physical processes that become important only when the gradients become large. However, **it is not generally true** that these higher order effects are dominated by dissipation — many other possibilities can occur. Below we consider one such alternative situation, which happens to be quite common

Remark 2.5 *The point made in the prior paragraph is very important, since shocks (as a resolution of the wave breaking caused by the kinematic wave steepening) are the answer* **only** *when the higher order effects are of a dissipative nature. One should be very careful about not introducing shocks into mathematical models of physical processes just because the models exhibit wave distortion and breaking. Many other behaviors are possible, some rather poorly understood. The answer to a mathematical breakdown in a model is almost never to be found purely by mathematical arguments; a careful look at the physical processes the equations attempt to model is a must when this happens.*

A possible, and quite frequent, alternative to dissipation is dispersion: namely, the wave speed is a non-trivial function of the wave number. The simplest instance of dispersion introduces a term proportional to the third space derivative of the solution in the equations. When coupled with the simplest kind of nonlinearity (quadratic), this gives rise to the **Korteweg-de Vries (KdV) equation.** The nondimensional form of the KdV equation is:

$$\boldsymbol{u_t + u\, u_x = \epsilon^2 u_{xxx},} \tag{2.11}$$

where $\boldsymbol{\epsilon > 0}$ is a parameter expressing the ratio of dispersion (different wavelengths moving at different speeds) to nonlinearity.

Remark 2.6 *The KdV equation describes (for example) the propagation of long, small (but finite) amplitude, waves in shallow water channels. The KdV equation is a "canonical" equation that arises in very many dispersive situations. That this should be so is relatively simple to see:*
First. *Consider a wave situation where the wave phase speed depends on the wave number in a nontrivial way:* $c_p = c_p(k)$. *Clearly,* c_p *should be an even function of* k, *since both* k *and* $-k$ *correspond to the same wavelength. Thus, for long waves (*k *small) one should be able to expand* c_p *in the form* $c_p = \alpha + \beta\, k^2 + \dots$ *(where* α *and* β *are constants). Furthermore, by changing coordinates into a moving frame, we can always assume* $\alpha = 0$.
Second. $c_p \approx \beta\, k^2$ *corresponds to a relationship between the wave number and the wave frequency of the form* $\omega = k\, c_p(k) \approx \beta\, k^3$, *which corresponds to the equation* $u_t = \beta u_{xxx}$. *Thus we see how a third order derivative arises as the simplest example of dispersive behavior for long waves.*

2.1.2 The problem to do.

Show that: **for the KdV equation above in (2.11), no shocks are possible.** To be precise, consider the non-trivial (i.e. $u \neq$ constant) traveling wave solutions of (2.11):

$$u = F(\zeta), \quad \text{where} \quad \zeta = \frac{x - V\, t - x_0}{\epsilon}, \tag{2.12}$$

$F = F(\zeta)$ is some smooth function, V is a constant (the wave speed), and x_0 is some arbitrary constant. Study all the solutions of this form such that (for some constants u_0 and u_1)

(a) $F(\zeta) \to u_0$ as $\zeta \to -\infty$, $F(\zeta) \to u_1$ as $\zeta \to \infty$, and F is not identically constant.

(b) The first and all the higher order derivatives of $F = F(\zeta)$ vanish as $\zeta \to \pm\infty$.

Then

$$\boxed{\textbf{Show that the conditions: } V < u_0 = u_1\textbf{, must apply.}} \tag{2.13}$$

These solutions cannot represent shock transitions, since the states at $\pm\infty$ are equal (therefore, no jump occurs). In fact, it is possible to write these solutions explicitly in terms of elementary functions (hyperbolic secant), but finding this explicit form is **optional.**

Remark 2.7 *The traveling waves described above (that you are supposed to analyze) are called* **solitary waves**, *because they consist of a single isolated disturbance that vanishes very quickly as* $x \to \pm\infty$. *Such waves are easy to see in shallow water situations, where disturbances of a wavelength much bigger than the depth are generated. For example, in early summer many lakes develop a thin layer (a few feet thick) of warm water over the colder (heavier) rest of the water. Waves on this top layer, with wavelengths much longer than its thickness, obey an equation similar to the KdV equation, also supporting solitary waves. These are easy to see (when they are generated by boats) as traveling bumps on the surface of the lake — these are "one-dimensional" bumps, that is: they decay only in the direction of propagation; the surface elevation is (basically) independent of the direction normal to the propagation direction (thus they look like "rolls", moving on the lake surface).*
Note that the solitary waves for (2.11) are actually "dips", not bumps. *In order to obtain "bumps", the sign of the dispersive term in (2.11) must be reversed. Namely, one must consider the equation*

$$v_t + v\, v_x = -\epsilon^2 v_{xxx}.$$

The traveling waves for this equation, and those for (2.11), are related. In fact: $u = f(x - V\,t)$ *is a traveling wave for (2.11), if and only if* $v = -f(x + V\,t)$ *is a traveling wave for the equation above.*

Hint 2.2 *First: substitute (2.12) into the KdV equation (2.11). This will yield a third order O.D.E. for the function* $F = F(\zeta)$. *Notice that the parameter* ϵ *is used in (2.12) to scale the wavelength of the traveling wave, precisely in the form needed to eliminate* ϵ *from the O.D.E. that* F *satisfies.*

Second: you should be able to integrate the third order O.D.E just obtained once, and reduce it to a second order O.D.E. — with some arbitrary integration constant. If you multiply this second order

O.D.E. by $\frac{dF}{d\zeta}$*, the result can again be integrated, so that you will end up with a first order O.D.E. for* F *— with two integration constants in it.*

What does "to integrate an O.D.E." mean?

To integrate an O.D.E., you must first write it in the form derivative (something) = 0*, from which you can conclude that* something = constant *— you have just "integrated" the O.D.E. once. Sometimes an O.D.E. cannot be written directly as the derivative of something, but it is still possible to do so by multiplying the equation by an appropriate* integrating factor*. This is what the multiplication by* $dF/d\zeta$ *in this hint will accomplish.*

Finally, consider the limits as $\zeta \to \pm\infty$ *of the O.D.E. you obtained in the previous steps. Using the assumed properties for* F*, you should now be able to obtain that* $u_0 = u_1$*. Showing that you also need* $V < u_0 = u_1$ *for a solution to exists is a bit trickier, and will require you to do some analysis of the equation — similar to the one used to study the traveling wave solutions of (2.6).*

2.1.3 First order O.D.E. review.

Here we review some facts about real valued solutions to O.D.E.'s of the form

$$\left(\frac{dy}{dx}\right)^2 = R(y)\,, \tag{2.14}$$

where R is a real valued, smooth function. We consider the case $\boldsymbol{R = (y-a)(y-b)^2}$ only, where a and b are constants. However, the same type of analysis can be used for R of general form, by considering the zeros of R — i.e.: the points y_* at which $R(y_*) = 0$. In particular, note that the case $R = c(y-a)(y-b)^2$, where $c > 0$ is a constant, can be reduced to this one by the simple scaling $x \to \sqrt{c}\,x$. We will also **assume that $a \neq b$**, since $a = b \implies y = a + \dfrac{4\delta}{(x-x_0)^2}$ (where x_0 is a constant and $\delta = 0$ or $\delta = 1$) and there is not much to analyze.

The **process below is based in the following idea, useful for any R:** once we know the behavior of the solutions of (2.14) near the zeros of R, and for $|y|$ large, the overall qualitative behavior is easy to ascertain. This is because the sign of $\dfrac{dy}{dx}$ can change only at the zeros of R, so that $y = y(x)$ must be monotone between zeros.

Imagine the solutions plotted in the $(x,\, y)$–plane — where we draw the horizontal lines $y = y_*$, for every zero of R. Now consider a solution in any horizontal strip between two such zeros: it will have to be monotone (increasing or decreasing) till it reaches one of the zeros. The key question is: can a zero be reached at a finite value of x? If not, then the solution will reach the zero **only** as either $x \to \infty$, or $x \to -\infty$. If yes, then we need to know what happens when the zero is reached, to continue the solution beyond it.

It turns out that only first order zeros can be reached at a finite value of x, and there the solution "bounces" back from the zero, turning from monotone increasing to monotone decreasing (or vice versa), with a sign change in $\dfrac{dy}{dx}$. Thus first order zeros of R are associated with local maximums (or minimums) of the solutions.

We must also study the solutions in the half planes between the largest (smallest) zero of R and ∞ (resp. $-\infty$). For this we need to know the solutions' behavior for $|y|$ large.

FIRST: It is clear that we **need** $y \geq a$. Else the right hand side in (2.14) is negative and the solution cannot be real valued — **except, possibly, for the trivial solution** $y \equiv b$ **(if** $b < a$**.)**

SECOND: Consider the behavior of the non-trivial solutions near the zeros of R (i.e.: exclude the solutions $y \equiv a$ and $y \equiv b$.)

- **For $\boldsymbol{y \approx a}$**, write $y = a + z(x)$, with $z > 0$ small. Then $(dz/dx)^2 = (a-b)^2\, z$ approximates the equation. This has the solutions $z = \dfrac{1}{4}(a-b)^2(x-x_0)^2$, where x_0 is a constant. Thus it is clear that the solutions of (2.14) will reach local minimums when y approaches a.
- **For $\boldsymbol{y \approx b}$**, write $y = b + z(x)$, with z small. Then $(dz/dx)^2 = (b-a)\, z^2$ approximates the equation. This has real solutions only if $b > a$, with $z = c\, e^{\kappa x}$, c a constant and $\kappa = \pm\sqrt{b-a}$. Thus the solutions of (2.14) can approach b in the limits $x \to \pm\infty$, but only if $b > a$.

THIRD: Use these results to analyze the real valued solutions of (2.14) in the two possible cases.

$\boxed{\textbf{Case } \boldsymbol{a > b.}}$ The solutions are real only if $y \geq a$ (except for the trivial solution $y \equiv b$). Then either $y \equiv a$ or $y > a$ somewhere. In this second case the solution has a minimum at some $x = x_0$ with $(dy/dx) > 0$ for $x > x_0$ and $(dy/dx) < 0$ for $x < x_0$. Away from x_0, the solution grows without bound. Eventually y becomes very large, $(dy/dx)^2 \approx y^3$, and the solutions blow up like $4/(x-x_*)^2$. **Thus the only bounded real solutions are the trivial ones $\boldsymbol{y \equiv a}$ and $\boldsymbol{y \equiv b}$.**

Case $b > a.$ If $y > b$ anywhere, the solution decays to $y = b$ as either $x \to \pm\infty$, with a singularity like $4/(x - x_*)^2$ at a finite $x = x_*$. This follows because either $(dy/dx) > 0$ or $(dy/dx) < 0$ and there is no way for the sign to change. On the other hand, if $a < y < b$, the solution decreases from $y = b$ at $x = -\infty$, to a minimum at some $x = x_0$ (where $y = a$) and then increases back to $y = b$ at $x = \infty$. **In this case nontrivial bounded solutions exist in the range $a \le y < b$.** These have limits $y = b$ at $x = \pm\infty$ and a single minimum (where $y = a$) at some finite $x = x_0$.

The analysis above depends only on the nature of the zeros of R. For $R = (y - a)(y - b)^2$ we can **solve the equation explicitly** and verify the results: Introduce $z = z(x)$ by $z^2 = y - a$ and write $\nu^2 = b - a$. Equation (2.14) — i.e.: $(y')^2 = (y - a)(y - b)^2$ — then becomes

$$4z^2(z')^2 = z^2(z^2 - \nu^2)^2 \iff 2z' = \pm(z^2 - \nu^2) \iff \ln\left(\frac{z - \nu}{z + \nu}\right) = \pm\nu x + \alpha,$$

where the prime denotes differentiation with respect to x and α is an arbitrary constant. Here we do not make any assumptions on the signs of $y - a$ and $b - a$: **thus z, ν and α need not be real.** With some further manipulation this yields, in terms of λ (defined by $\lambda^2 = -\exp(\pm\nu x + \alpha)$),

$$z = \nu\frac{1 - \lambda^2}{1 + \lambda^2} \iff \nu^2 - z^2 = \nu^2\left(\frac{2}{\lambda + \lambda^{-1}}\right)^2 \iff y = b - \nu^2\left(\frac{2}{\lambda + \lambda^{-1}}\right)^2.$$

Finally we can write, being now careful with keeping things real:

Case $a > b.$ Let $\mu > 0$ be defined by $\mu^2 = a - b$. Then the nontrivial real solutions are

$$y = b + \mu^2 \sec^2\left(\frac{1}{2}\mu\,(x - x_0)\right), \tag{2.15}$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$.) The relationship with the constants defined earlier is $\nu = \pm i\mu$ and $\alpha \pm \nu x_0 = i\pi$. See the left frame in figure 2.2.

Case $b > a.$ Let $\mu > 0$ be defined by $\mu^2 = b - a$. Then the nontrivial real solutions are

$$y = b - \mu^2\,\mathrm{sech}^2\left(\frac{1}{2}\mu\,(x - x_0)\right) \quad \text{and} \quad y = b + \mu^2\,\mathrm{cosech}^2\left(\frac{1}{2}\mu\,(x - x_0)\right), \tag{2.16}$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$.) The relationship with the constants defined earlier is $\nu = \pm\mu$ and either $\alpha \pm \nu x_0 = i\pi$ or $\alpha \pm \nu x_0 = 0$. See the right frame in figure 2.2.

Next we briefly summarize the **situation for more general forms of** R — which can be analyzed with the same techniques we used here. For non-trivial real valued solutions:

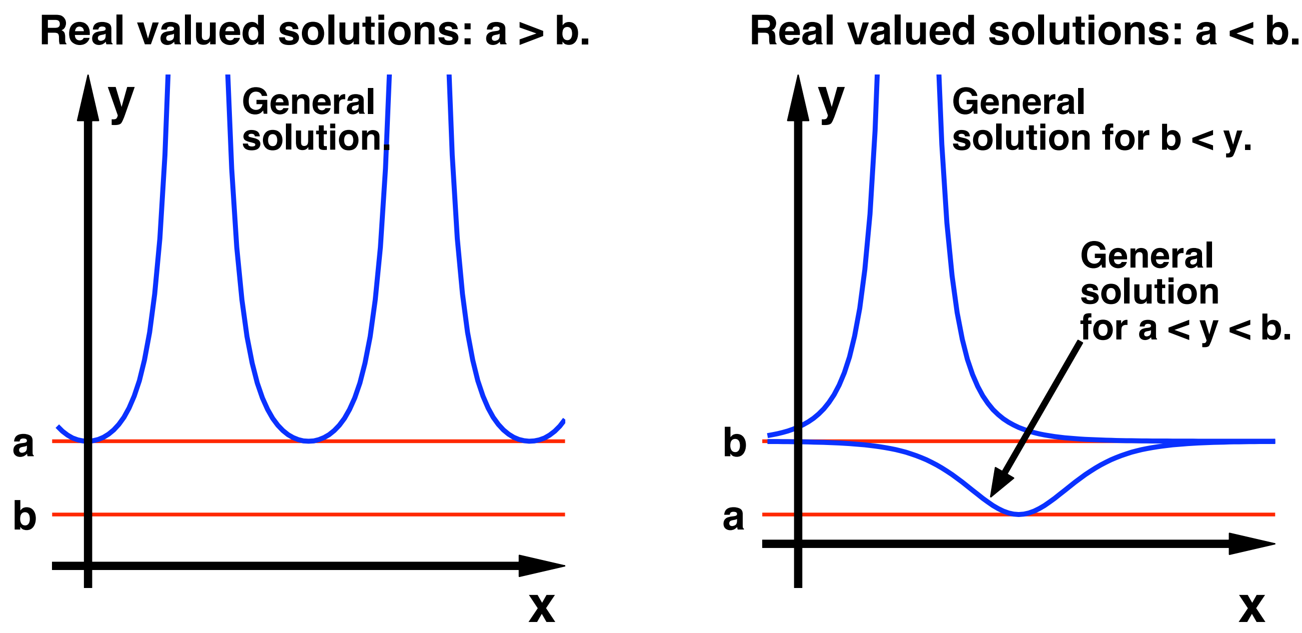

Figure 2.2: Real valued solutions of equation (2.14), with $R = (y-a)(y-b)^2$.

- Left frame: case $a > b$. The non-trivial solutions are periodic, with period $T = 2\pi/\sqrt{a-b}$.
- Right frame: case $a < b$. All the solutions satisfy $y \to b$ as $|x| \to \infty$.

1. At a first order zero of R, the solutions will either achieve a local minimum — if $dR/dy > 0$ there, or a local maximum — if $dR/dy < 0$ there. In other words, solutions can achieve values that correspond to simple zeros of R at some finite value of x, where they will have a local maximum or a local minimum. But these values cannot occur in the limits $x \to \pm\infty$.

2. By contrast, solutions cannot achieve values that correspond to higher (bigger than one) order zeros of R, at any finite value of x. Such values can be achieved only in the limits where $x \to \pm\infty$. For double zeros of R, the approach to the value (as $x \to \pm\infty$) is exponential. For zeros of order higher than two, the approach is algebraic .

3. Periodic oscillatory solutions occur in the regions between simple zeros, where $R > 0$. This situation does not occur in the example treated earlier, which has a single simple zero.

4. Single bump (or dip) solutions occur in the regions, comprised between a simple and a double (or higher) order zero, where $R > 0$. The right frame in figure 2.2 shows an example of this.

5. The behavior of the solutions in regions where $R > 0$, comprised between a zero of R and $\pm\infty$, depends on the behavior of $R(y)$ for large values of y. If this leads to the formation of singularities (e.g.: $R = R(y)$ grows faster than linear as $y \to \pm\infty$), then the solutions are as in figure 2.2: either a periodic array of singularities (for a simple zero, left frame) or a single singularity — with decay to the value of y at the zero of R — as $x \to \pm\infty$ (for a double or higher order zero, right frame.)

2.2 Statement: BuHe01. Traveling wave solutions and shocks.

Imagine that someone tells you that

$$c_t + c\, c_x = \nu\, c_{xt}, \tag{2.17}$$

where $\boldsymbol{\nu > 0}$ is "small",[1] is a model for traffic flow (c here is the wave velocity, not the car density).

> **Does this model have acceptable solutions of the form $\boldsymbol{c = F(x - U\,t)}$, where where $\boldsymbol{U}$ is a constant, and $\boldsymbol{c}$ approaches constant values as $\boldsymbol{z = \dfrac{x - U\,t}{\nu} \to \pm\infty}$?**

These solutions should represent shock transitions in the flow. **If there are such solutions, find $\boldsymbol{F}$ and $\boldsymbol{U}$. Then check to see if these solutions are compatible with the Rankine-Hugoniot jump conditions, and with the Entropy condition.**

Remark 2.8 *An attempt at "justifying" (2.17) could go as follows:*

It is not unreasonable to assume that the drivers not only respond to the local traffic density, but its rate of change as well. A simple way to model this is to write $$q = Q(\rho) - \nu\rho_t, \tag{2.18}$$ *for the flow rate $q = q(x,\, t)$, where $\nu > 0$ is a constant characterizing the drivers response to the local rate of change in the density — the reason ν should be positive, is that the normal driver's reaction to a density increase should be to slow down.*

Substituting (2.18) into the equation for conservation of cars, yields:

$$\rho_t + c(\rho)\,\rho_x = \nu\rho_{tx}\,, \tag{2.19}$$

where $c = \dfrac{dQ}{d\rho}$. In particular, when Q is a quadratic function of ρ, c is a linear function of ρ and (2.19) is equivalent to (2.17).

[1]Note that ν has dimensions of inverse length, so small means compared with some appropriate length scale.

2.3 Statement: TFPb06. Damped kinematic equation.

Consider the following initial value problem

$$p(x,\,0) = \begin{cases} 0 & \text{for} \quad 0 < x\,, \\ 1+x & \text{for} \quad -1 \le x < 0\,, \\ 0 & \text{for} \quad x \le -1\,, \end{cases} \tag{2.20}$$

for the quasi-linear, first order, partial differential equation

$$p_t - p\,p_x = -a\,p\,, \tag{2.21}$$

where $\boldsymbol{a > 0}$ is a constant, $-\infty < x < \infty$, and $t \ge 0$.

a. There exists a value $\boldsymbol{a_c}$, such that: *For $a \ge a_c$ no shocks are needed in the solution of (2.20 – 2.21), while for $a < a_c$ they are needed.* **What is the value $\boldsymbol{a_c}$?**

Hint: solve for the characteristic curves, and find when they cross, or not. What happens with the characteristics that start at the origin, where the initial conditions have a discontinuity?

b. Compute (explicitly) the solution to (2.20 – 2.21) for $\boldsymbol{a \ge a_c}$, and **calculate** the limit

$$\lim_{t\to\infty} e^{a\,t} p(x,\,t) = \psi(x)\,. \tag{2.22}$$

c. Sketch a plot of $\boldsymbol{\psi = \psi(x)}$ in (2.22).

d. What is the behavior of the solution, when $\boldsymbol{a \ge a_c}$, for large times?

Hint. Use the result from part **b.**

Can you explain how is it that the damping term is preventing the solution from breaking and developing shocks? *A comparison of the answer here with that of part d in problem TFPb05 may prove useful in arriving at am answer.*

e. Sketch the characteristics in an x–t diagram, for the cases $\boldsymbol{a > a_c}$ and $\boldsymbol{a = a_c}$.

2.4 Statement: FPb13. Expansion fans for quadratic flows.

Consider the special example where the traffic flow rate is given by a quadratic function $q = q(\rho)$. In this case the traffic flow equations can be written in the following non-dimensional form

$$\rho_t + c\rho_x = 0\,, \quad \text{with} \quad c = 4(1-2\rho), \tag{2.23}$$

corresponding to a flow rate $q = 4\,\rho\,(1 - \rho)$. This is done by using the jamming density as the unit for car density, the maximum flow rate (road capacity) as the flow rate unit, some length scale given by the initial conditions, and a corresponding time scale.

As explained in the lectures, discontinuities in the initial conditions or boundary conditions — where the traffic density ρ increases with x, or decreases with t, as the discontinuity is crossed — give rise to expansion fan regions for the characteristics. In these regions the solution has the form

$$\rho = \frac{1}{2} - \frac{1}{8}\,\frac{x - x_0}{t - t_0}, \tag{2.24}$$

where x_0 and t_0 are constants (the expansion fan of characteristics is centered at $(x,\, t) = (x_0,\, t_0)$ in space time).

A. Show directly that (2.24) solves (2.23). Notice that, **since equation (2.23) is invariant under space and time translations** — i.e.: it does not matter where the origin of coordinates is — **you can assume that both x_0 and t_0 vanish in this calculation.**

B. Find the general solution for a car trajectory in a region of space time where (2.24) applies. That is, solve the equation:

$$\frac{dx}{dt} = u(\rho(x,\, t))\,, \tag{2.25}$$

where $u = u(\rho)$ is the car velocity. Again: you can assume that both x_0 and t_0 vanish.

THE END.