18.311 — MIT (Spring 2010)

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Problem Set # 04.

Due: Tuesday March 16. Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Generic HINTS:

1. *When $q = q(\rho)$ is quadratic, the shock speed is the average of the characteristic speeds immediately to the right and left of the shock. For many problems this simplifies the algebra.*
2. *The general solution of a linear o.d.e. with a forcing term can be written as the sum of a particular solution, plus the general solution to the homogeneous problem.*
3. *When solving the o.d.e. for the shock path, as given by the Rankine-Hugoniot jump conditions, beware of the fact that many of the solutions in the problem sets are given by different formulas in different regions. Hence, as a shock enters a different region, the o.d.e. changes. Keep track of this. Example: a shock starts with constant velocity (states on each side are constant) and switches to variable velocity (enters a region with a rarefaction fan on one side).*
4. *Remember that the characteristics are the curves along which information propagates. A situation where the solution along a characteristic is determined backwards in time is* not physically meaningful, *as it violates causality.* Make sure that your solutions satisfy causality!
5. Always: check that your answers are sensible. *If it seems as if they predict something that contradicts physical observation, then something is probably wrong!*

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1 Regular Problems.

1.1 Statement: Haberman problem 77.01.

If $u = u_{\mathsf{max}}(1 - \rho/\rho_{\mathsf{max}})$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? Simplify the expression as much as possible. Show that the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1.

1.2 Statement: Haberman problem 77.04..

Show $[\,\rho^2\,] \neq [\,\rho\,]^2$.

1.3 Statement: Haberman problem 77.05.

Suppose that, instead of $u = U(\rho)$, the car velocity u is

$$u = U(\rho) - \frac{\nu}{\rho}\,\rho_x, \tag{1.1}$$

where ν is a constant.

(a) What sign should $\boldsymbol{\nu}$ have for this expression to be physically reasonable?

(b) What equation now describes conservation of cars?

(c) Assume that $U(\rho) = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$. Show that

$$\rho_t + u_{\mathsf{max}}\left(1 - \frac{2\,\rho}{\rho_{\mathsf{max}}}\right)\rho_x = \nu\,\rho_{xx}, \tag{1.2}$$

called **Burger's equation.**

1.4 Statement: Haberman problem 78.08.

Determine the traffic density on a semi–infinity ($x > 0$) highway for which the density at the entrance is

$$\rho(0,\,t) = \begin{cases} \rho_1 & \text{for } \ 0 < t < \tau, \\ \rho_0 & \text{for } \ \tau < t, \end{cases} \tag{1.3}$$

where $\tau > 0$ is constant, and the initial density is uniform along the highway — assume that $\rho(x,\,0) = \rho_0$, for $x > 0$. Furthermore, assume that ρ_1 is lighter traffic than ρ_0, and that both are light traffic; in fact assume that $u(\rho) = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$ and that $\rho_1 < \rho_0 < \rho_{\mathsf{max}}/2$. Sketch the density at various values of time.

1.5 Statement: TFPa11. Longest queue through a light.

A traffic signal (at $x = 0$) is green for $0 \le t \le T$, and red for all other times. If $\rho(x,\,0) = \rho_j$ for $x \le 0$, $\rho(x,\,0) = 0$ for $x > 0$, and $q = (4\,q_m/\rho_j^2)\,\rho\,(\rho_j - \rho)$, determine the trajectory of the last car to make the light. What is the longest traffic queue that can pass through the intersection during the green light?

1.6 Statement: TFPa17. Shock interaction with a traffic light.

At time $t = 0$, the traffic pattern on a long highway consists of two sections of constant concentration, joined by a shock which moves in the positive x direction, as shown in figure 1.1. If a traffic light at $x = 0$ turns (at time $t = 0$) and remains red, describe the resultant motion. Let the position of the shock at time $t = 0$ be given by $x = -L < 0$, and assume that $q = \dfrac{4\,q_m}{\rho_j^2}\,\rho\,(\rho_j - \rho)$ — with $0 < \rho_0 < \rho_1 < (1/2)\,\rho_j = \rho_m$.

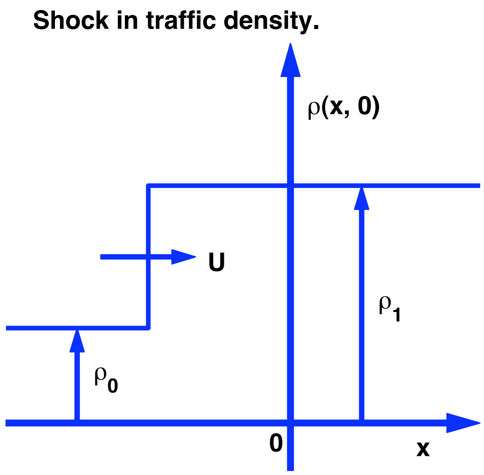

Figure 1.1: TFPa17: Shock in traffic density.

1.7 Statement: TFPa24.
Semi-linear first order equation and characteristics.

Consider the equation

$$x^2\,\psi_x + x\,y\,\psi_y = \psi^2\,, \tag{1.4}$$

subject to $\boldsymbol{\psi = 1}$ on the curve $\boldsymbol{\Gamma}$ given by $\boldsymbol{x = y^2}$. This is a semi-linear problem that can be written in terms of characteristics.

A. Compute the characteristic curves that cross the curve Γ, as follows: (i) Parameterize the curve Γ, say: $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{-\infty < \xi < \infty}$. (ii) Write the o.d.e.'s for the characteristic curves, in terms of some parameter (say, s) along each curve. (iii) Solve the o.d.e.'s for the characteristics, with the condition that $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{s = 0}$.

B. Describe what type of curves, in the x-y plane, are the characteristics. Which region of the plane do these curves cover? What happens with the characteristic corresponding to $\xi = 0$?

C. Solve the o.d.e. that ψ satisfies along each characteristic. Eliminate the parameters ξ and s in terms of x and y, and write an explicit formula for the solution $\psi = \psi(x, y)$ to (1.4).

D. Where is the solution ψ defined? **Hint:** *be careful with your answer here!*

2 Special Problems.

2.1 Statement: TFPb14. Check if discontinuities are allowed shocks.

Consider the conservation equation $\boldsymbol{\rho_t + q_x = 0}$ (for the conserved density ρ) with various choices for the flow rate $q = q(\rho)$ — as given below. Assume that the underlying physical processes behind this conservation equation lead to the formation of shocks — as a resolution of the wave–breaking caused by the crossing of the characteristics.

In each case a *candidate discontinuous solution* is proposed, of the the form:

$$\rho(x,t) = \begin{cases} \rho_l & \text{for} \quad x < 0 \quad \text{and all} \quad t > 0, \\ \rho_r & \text{for} \quad x > 0 \quad \text{and all} \quad t > 0, \end{cases} \tag{2.5}$$

where $\boldsymbol{\rho_l}$ and $\boldsymbol{\rho_l}$ are constants. **Notation:** $q_l = q(\rho_l)$, $q_r = q(\rho_r)$, $c_l = c(\rho_l)$, and $c_r = c(\rho_r)$, where $c = c(\rho) = \dfrac{dq}{d\rho}$ is the characteristic speed.

Check if the given "candidate solution" are actually solutions. Justify your answers: If something is a solution, verify this and, if it is not, say why not.

A. $\boldsymbol{q = \rho(2-\rho)}$, with $\rho_l = 0.5$, $\rho_r = 1.5$, $q_l = q_r = 0.75$, $c_l = 1$, and $c_r = -1$.

B. $\boldsymbol{q = \rho(2-\rho)}$, with $\rho_l = 0$, $\rho_r = 1$, $q_l = 0$, $q_r = 1$, $c_l = 2$, and $c_r = 0$.

C. $\boldsymbol{q = \rho^2}$, with $\rho_l = -1$, $\rho_r = 1$, $q_l = q_r = 1$, $c_l = -2$, and $c_r = 2$.

D. $\boldsymbol{q = 1 - \rho^2}$, with $\rho_l = -1$, $\rho_r = 1$, $q_l = q_r = 0$, $c_l = 2$, and $c_r = -2$.

E. $\boldsymbol{q = \rho(2-\rho)}$, with $\rho_l = 1.25$, $\rho_r = 0.75$, $q_l = q_r = 0.9375$, $c_l = -0.5$, and $c_r = 0.5$.

2.2 Statement: TFPb15. Traffic lights at the ends of a tunnel.

Consider a tunnel in a road, governed by the traffic flow equation $\rho_t + q_x = 0$ applies. Assume that the following situation occurs (we use non-dimensional variables)

A. The flux is given by $q = \rho\,(2-\rho)$. Thus $c = 2\,(1-\rho)$ — wave speed, $u = (2-\rho)$ — flow velocity, $\rho_J = 2$ — jamming density, and $q_M = 1$ — road capacity, occurring for $\rho_M = 1$.

B. The tunnel is located at $0 \leq x \leq 1$.

C. There are traffic lights at both tunnel ends: one light at $x = 0$, and another one at $x = 1$.

D. For $t < 0$ both lights are green and the flow is uniform, at the road capacity $\rho \equiv \rho_M = 1$.

E. At time $t = 0$ **both** traffic lights go simultaneously red, and stay red from then on.

Solve for the traffic flow density ρ inside the tunnel for all times $t \geq 0$. An explicit (very explicit) solution is required. There are two very "special" waves that will arise at the positions of the traffic lights; what are they (i.e.: physically, what do they mean)?

THE END.