18.311 — MIT (Spring 2010)

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Problem Set # 03.

Due: Tuesday March 9. Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Generic Hints:

1. *When $q = q(\rho)$ is quadratic, the shock speed is the average of the characteristic speeds immediately to the right and left of the shock. For some of the problems this simplifies the algebra. This will be shown in the lectures, though note that problem 77.03 below is as instance of this.*
2. *The general solution of a linear o.d.e. with a forcing term can be written as the sum of a particular solution, plus the general solution to the homogeneous problem.*
3. *When solving the o.d.e. for the shock path, as given by the Rankine-Hugoniot jump conditions, beware of the fact that many of the solutions in the problem sets will be given by different formulas in different regions. Hence you have to consider that, as the shock enters a different region, the o.d.e. changes. Be careful to keep track of this. A shock may start with constant velocity (if the states on each side are constant) and then switch to variable velocity (say, it enters a region where it has a rarefaction fan on one side).*
4. *Remember that the characteristics are the curves along which information propagates. Hence, a situation where the solution along a characteristic is determined backwards in time is not physically meaningful, as it violates causality.* Make sure that your solutions satisfy causality!
5. Always: check that your answers are sensible. *If they seem as if they predict something that contradicts physical observation, then something is probably wrong!*

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1 Regular Problems.

1.1 Statement: Haberman problem 70.03.

Assuming nearly uniform, but heavy, traffic, show that in general it is impossible to prescribe the traffic flow at the entrance to a semi–infinite highway. In this situation, what might happen to cars waiting to enter the highway?

1.2 Statement: Haberman problem 72.01.

Show that if $u = u(\rho)$ is determined by braking distance theory (see exercise 61.2), then the waiting time per car after a traffic light turns green is the same as the human reaction time for braking.

1.3 Statement: Haberman problem 73.01.

Assume that the traffic density is initially given by

$$\rho(x,\,0) = \begin{cases} \rho_{\mathsf{max}} & \text{for } \; x < 0, \\ \frac{1}{2}\,\rho_{\mathsf{max}} & \text{for } \; 0 < x < a, \\ 0 & \text{for } \; a < x, \end{cases} \tag{1.1}$$

where $a > 0$ is some fixed length, and that the car flow velocity is related to the car density by

$$u = u_{\mathsf{max}}\left(1 - \frac{\rho}{\rho_{\mathsf{max}}}\right) \implies q = u_{\mathsf{max}}\left(1 - \frac{\rho}{\rho_{\mathsf{max}}}\right)\rho \implies c = u_{\mathsf{max}}\left(1 - 2\,\frac{\rho}{\rho_{\mathsf{max}}}\right). \tag{1.2}$$

Sketch the initial density. Determine and sketch the density at all later times.

1.4 Statement: Haberman problem 73.02 and 03.

Assume that the car flow velocity is related to the car density by:

$$u = u_m\left(1 - \frac{\rho}{\rho_j}\right) \implies q = u_m\left(1 - \frac{\rho}{\rho_j}\right)\rho \implies c = u_m\left(1 - 2\,\frac{\rho}{\rho_j}\right), \tag{1.3}$$

where ρ_j is the jamming density and u_m is the car speed limit. Consider now the **red light turns green** problem and:

First: Calculate the maximum acceleration of a car which starts approximately one car length behind a traffic light (i.e. $x(0) = -1/\rho_j$).

Second: Calculate the velocity of a car at the moment it starts moving behind a light.

1.5 Statement: Haberman problem 74.01.

Assume that $u(\rho) = u_m\,(1 - \rho/\rho_j)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x < 0, \\ \rho_0\,(L - x)/L & \text{for } \; 0 \le x \le L, \\ 0 & \text{for } \; L < x, \end{cases} \tag{1.4}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, determine and sketch $\rho(x,\, t)$.

1.6 Statement: Haberman problem 74.02.

Assume that $u(\rho) = u_m\,(1 - \rho^2/\rho_j^2)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\, 0) = \begin{cases} \rho_0 & \text{for } \ x < 0, \\ \rho_0\,(L - x)/L & \text{for } \ 0 < x < L, \\ 0 & \text{for } \ L < x, \end{cases} \tag{1.5}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, **determine and sketch** $\rho(x,\, t)$**.**

1.7 Statement: Haberman problem 74.03.

Consider the (non-dimensionalized) partial differential equation

$$\rho_t - \rho^2\,\rho_x = 0\,, \quad -\infty < x < \infty \ \text{ and } \ t > 0. \tag{1.6}$$

(a) Why can't this equation model a traffic flow problem?

(b) Solve this P.D.E. by the method of characteristics, subject to the initial conditions:

$$\rho(x,\, 0) = \begin{cases} 1 & \text{for } \ x < 0, \\ 1 - x & \text{for } \ 0 \leq x \leq 1, \\ 0 & \text{for } \ 1 < x. \end{cases} \tag{1.7}$$

1.8 Statement: Haberman problem 77.03.

A **weak shock** is a shock in which the shock strength (the difference in densities) is small. For a weak shock, show that the shock velocity is *approximately* the average of the density wave velocities associated with the two densities. [**Hint:** Use Taylor series methods.]

1.9 Statement: Haberman problem 78.01.

Suppose that the initial traffic density is

$$\rho(x,\, 0) = \begin{cases} \rho_0 & \text{for } \ x < 0, \\ \rho_1 & \text{for } \ x > 0, \end{cases} \tag{1.8}$$

where ρ_0 and ρ_1 are constants. Consider the two cases, $\rho_0 < \rho_1$, and $\rho_0 > \rho_1$. For which of the preceding cases is a density shock necessary? Briefly explain.

1.10 Statement: Haberman problem 78.03.

Assume that $u = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$ and at $t = 0$ the traffic density is

$$\rho(x,\,0) = \begin{cases} (1/3)\,\rho_{\mathsf{max}} & \text{for } \ x < 0, \\ (2/3)\,\rho_{\mathsf{max}} & \text{for } \ x > 0. \end{cases} \tag{1.9}$$

Why does the density not change in time?

2 Special Problems.

No special problems with this set.

THE END.