18.311 — MIT (Spring 2010)

Rodolfo R. Rosales (MIT, Math. Dept., 2-337, Cambridge, MA 02139).

February 23, 2010.

Problem Set # 02.

Due: Tuesday March 2.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems. 2

1.1 Statement: Haberman problem 61.03 2

Formula for car's acceleration. 2

1.2 Statement: Haberman problem 63.02 2

State laws and road capacity. 2

1.3 Statement: Haberman problem 63.06 2

Formula for car acceleration. 2

1.4 Statement: Haberman problem 63.07 2

Find. 2

1.5 Statement: Haberman problem 71.01 3

Experimental traffic flow function. 3

1.6 Statement: Haberman problem 71.02 3

Density wave velocity relative to a moving car. 3

1.7 Statement: Haberman problem 71.04 3

p.d.e. satisfied by the wave velocity. 3

1.8 Statement: Linear 1st order PDE problem 02 4

Solve a linear first order p.d.e. using characteristics. 4

2 Special Problems. **4**
2.1 Statement: Linear 1st order PDE problem 03 4
Solve a linear first order p.d.e. using characteristics. 4

List of Figures

1 Regular Problems.

1.1 Statement: Haberman problem 61.03.

Assume that a velocity field, $u = u(x, t)$ exists. Show that the acceleration of an individual car is given by

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}. \tag{1.1}$$

1.2 Statement: Haberman problem 63.02.

If cars obey state laws on following distances (refer to exercise 61.01), what is the road capacity if the speed limit is 50 m.p.h (80 k.p.h.)? At what density and velocity does this maximum occur? Will increasing the speed limit increase the road's capacity?

1.3 Statement: Haberman problem 63.06.

Assume that $u = u(\rho)$. If α denotes the car's acceleration, show that

$$\alpha = -\rho\,\frac{du}{d\rho}\,\frac{\partial u}{\partial x}. \tag{1.2}$$

Is the minus sign here reasonable?

1.4 Statement: Haberman problem 63.07.

Consider exercise 61.03. Suppose that the drivers accelerate in such a fashion that

$$\alpha = u_t + u\,u_x = -\frac{a^2}{\rho}\rho_x, \quad \text{where } a > 0 \text{ is a constant.} \tag{1.3}$$

(a) Physically interpret this situation.

(b) If u only depends on ρ, and the equation for conservation of cars is valid, show that

$$\frac{du}{d\rho} = -\frac{a}{\rho}. \tag{1.4}$$

(c) Solve the differential equation in part **(b)**, subject to the condition that $u(\rho_{\mathsf{max}}) = 0$. The resulting flow–density curve fits quite well to the Lincoln Tunnel data.

(d) Show that a is the velocity that corresponds to the road's capacity.

(e) Discuss objections to the theory for small densities.

1.5 Statement: Haberman problem 71.01.

Experiments in the Lincoln Tunnel (combined with theoretical work discussed in exercise 63.07) suggests that the traffic flow function is, approximately,

$$q(\rho) = a\,\rho\,\left(\ln(\rho_{\mathsf{max}}) - \ln(\rho)\right), \tag{1.5}$$

where a and ρ_{max} are known constants. Suppose that the initial density $\rho(x,\,0)$ varies linearly from bumper-to-bumper traffic (behind $x = -x_0 < 0$) to no traffic (ahead of $x = 0$), as sketched in figure 71-6. Two hours later, where does $\rho = \dfrac{1}{2}\,\rho_{\mathsf{max}}$?

1.6 Statement: Haberman problem 71.02.

Referring to the theoretical flow-density relationship of exercise 71.01, show that the density wave velocity relative to a moving car is the same constant no matter what the density.

1.7 Statement: Haberman problem 71.04.

Consider the p.d.e. $\rho_t + q_x = \rho_t + c(\rho)\,\rho_x = 0$, where $q = q(\rho)$ and $c = c(\rho) = \dfrac{dq}{d\rho}(\rho)$. Assume that $\rho = \rho(x,\,t)$ is a smooth solution of the equation, and let $c = c(x,\,t) = c(\rho)$. **Show that:**

$$c_t + c\,c_x = 0. \tag{1.6}$$

1.8 Statement: Linear 1st order PDE problem 02.

Consider the following problem

$$x\,u_x + y\,u_y = 1 + y^2, \qquad \text{with} \qquad u(x,\,1) = 1 + x \quad \text{for} \quad -\infty < x < \infty. \tag{1.7}$$

Part 1. Use the method of characteristics to solve this problem. Write the solution $\boldsymbol{u = u(x,\,y)}$ **(explicitly!)** as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\boldsymbol{y > 0}$. **Hint.** *Write the characteristic equations:* $\dfrac{dx}{ds} = \ldots$, $\dfrac{dy}{ds} = \ldots$, *and* $\dfrac{du}{ds} = \ldots$. *Then solve these equations using the initial data (for $s = 0$) $x = \tau$, $y = 1$, and $u = 1 + \tau$, for $-\infty < \tau < \infty$. Finally, eliminate s and τ, to get u as a function of x and y.*

Part 2. Explain why $\boldsymbol{u = u(x,\,y)}$ is not determined by the problem above for $\boldsymbol{y \leq 0}$ (you may use a diagram). **Hint.** *Draw, in the x-y plane, the characteristic curves computed in part 1.*

2 Special Problems.

2.1 Statement: Linear 1st order PDE problem 03.

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{2.8}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to solve this problem. Explicitly write the solution $\boldsymbol{u = u(x,\,y)}$ as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, using $\boldsymbol{f}$. **Hint.** *Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$, for $-\infty < \tau < \infty$. Finally, eliminate τ, to get u as a function of x and y.*

Part 2. In which part of the $(x,\,y)$ plane is the solution determined? **Hint.** *Draw, in the x-y plane, the characteristic curves computed in part 1.*

Part 3. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

THE END.