18.311 — MIT (Spring 2010)

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Problem Set # 01.

Due: Tuesday February 23.

Turn it in before 3:30 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 58.03.

Consider the process of calculating the traffic density ρ, from knowledge of the car positions, as follows:

$$\left.\begin{array}{l}\text{At any given location } x_0 \text{ in space (for a given time } t_0\text{), take the road} \\ \text{interval given by } |x - x_0| \leq 0.5\,\Delta x \text{ (for some “small” length } \Delta x\text{) and} \\ \text{count the number of cars } N \text{ in this interval. Then: } \boldsymbol{\rho(x_0,\, t_0) = N/\Delta x}.\end{array}\right\} \tag{1.1}$$

This definition depends on the choice of Δx, but the idea is to take Δx small, but large enough that N is “large”. In this case, provided that the number of cars per unit length does not change too fast (neither in space, nor in time), a “good” measurement of the density can be obtained. This exercise will guide you through the verification of this.

First, notice that the defined density has a discontinuity every time an additional car is included (such discontinuities appear as we either change Δx, or move in space and/or time).

- **(a)** What is the jump $\Delta\rho$ in density (i.e.: the magnitude of the discontinuity)?
- **(b)** If cars are equally spaced (say: $\rho_c = 100$ per mile), how long must the measuring interval be such that the discontinuities in density are less than 5 percent of the density? Namely: so that $\Delta\rho < \rho/20$. How many cars are then contained in this measuring distance?
- **(c)** Generalize your results to a roadway with a constant density ρ_c per mile.

From item **(c)** a formula of the form $\Delta x > \Delta_c$ will follow — where Δ_c will depend on what size of error (i.e.: the jumps in $\Delta\rho/\rho$) you are willing to tolerate (in this problem $1/20$). Let then L be a typical length scale over which the traffic road conditions change. Then the continuum approximation, in which we replace the cars by a density, will make sense as long as $\Delta_c \ll L$.

1.2 Statement: Haberman problem 59.01.

For traffic flow moving at velocity $u = 10$ m.p.h (16 k.p.h), such that cars are one car length behind each other, what is the traffic flow q?

Let L be the car length. A typical car length is $L = 16$ ft. (4.85 m).

1.3 Statement: Haberman problem 60.03.

(a) Without any mathematics, explain why $\displaystyle\int_{\boldsymbol{a(t)}}^{\boldsymbol{b(t)}} \boldsymbol{\rho(x,\, t)\, dx}$ is constant if a and b (not equal to each other) are moving with the traffic.

(b) Using part **(a)**, re-derive the equation

$$\frac{d}{dt}\int_a^b \rho\,(x,\, t)\, dx = q(a,\, t) - q(b,\, t), \tag{1.2}$$

where $a < b$ are any two points in the road.

(c) Assuming $\dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x}(\rho\, u)$, verify mathematically that part **(a)** is valid.

1.4 Statement: Haberman problem 60.04.

If the traffic flow is increasing as x increases $(\dfrac{\partial q}{\partial x} > 0)$, explain physically why the density must be decreasing in time $(\dfrac{\partial \rho}{\partial t} < 0)$.

1.5 Statement: Haberman problems 61.01 & 61.02.

Many state laws say that, for each 10 m.p.h (16 k.p.h) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 feet (5 meters)). There is another law that gives a maximum speed limit (assume that this is 50 m.p.h (80 k.p.h)). **Find the flow of cars as a function of density, $q = q(\rho)$.**

The state laws on following distances stated in the prior paragraph, were developed in order to prescribe spacing between cars such that rear-end collisions could be avoided, as follows:

(a) Assume that the car immediately ahead stops instantaneously. How far would the car following at $\boldsymbol{u}$ m.p.h travel, if

(1) the driver's reaction time was $\boldsymbol{\tau}$, and

(2) after a $\boldsymbol{\tau}$ delay, the driver decelerated at constant maximum deceleration $\boldsymbol{\alpha}$?

(b) The calculation in part **(a)** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate $\boldsymbol{\alpha}$, but the driver following still takes time $\boldsymbol{\tau}$ to react. How far back does a car have to be, traveling at $\boldsymbol{u}$ m.p.h, in order to prevent a rear-end collision?

(c) Show that the law described in the first paragraph of this problem corresponds to part **(b)**, if the human reaction time is about 1 second and the length of a car is about 16 feet (5 meters).

1.6 Statement: Haberman problem 67.04.

Consider the equation

$$(\rho_1)_t + c\,(\rho_1)_x = 0. \tag{1.3}$$

Suppose that we observe ρ_1 in a coordinate system moving at velocity v. Show that

$$(\rho_1)_t + (c - v)\,(\rho_1)_x = 0. \tag{1.4}$$

Does the car density ρ stay constant moving at the car velocity?

1.7 Statement: Haberman problem 69.01.

Show that for an *observer moving with the traffic*, the rate of change of the measured density is

$$\frac{d\rho}{dt} = (u - c(\rho))\,\rho_x, \tag{1.5}$$

where $c = \dfrac{dq}{d\rho}$.

2 Special Problems.

2.1 Statement: Dimensional analysis and diffusion speed.

In the lectures it was shown that if $\Theta = \Theta(\vec{x},\, t)$ denotes the concentration of salt in water (e.g.: grams per liter) then, **assuming that there is no motion** by the water

$$\Theta_t = \nu\,\Delta\,\Theta, \tag{2.6}$$

where ν is the diffusion coefficient — which we assume here to be constant[1] — and Δ is the *Laplacian operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The **same equation** (with a different value of ν) applies if, for example, Θ denotes sugar concentration, or the concentration of some coloring (e.g. ink).

Under the conditions where (2.6) applies, imagine that you inject a very tiny blob of ink inside the liquid.[2] Then the size of the blob will start increasing with time (due to the ink's diffusion). The blob's edge will cease to be sharp as time goes on, but here we will simplify things and assume that they remain sharp enough during the course of the experiment.

Make the approximation that, at time $t = 0$ (when you start the experiment) the blob is just a point. Then, using qualitative and physical arguments only **(no solution of the equation)**

1. Argue that the shape of the blob is a sphere for $t > 0$.

2. Find a formula for the radius of the blob $R = R(t)$ as a function of time. There is a numerical multiplicative constant α, as in $R = \alpha\, f(t)$, which you will not be able to determine (without solving the p.d.e.), but you should be able to get $f(t)$.

Typical values for diffusion coefficients, as in the examples above, are $\boldsymbol{\nu = \gamma\, 10^{-5}\ \mathrm{cm}^2/\mathrm{sec}}$, where γ ranges[3] between 1/2 and 2. **Assume that $\boldsymbol{\alpha = \gamma = 1}$.** Then

3. What is the radius of the blob when $t = 1$ minute?

4. At what time does $R = 5$ cm?

These numbers should give you an idea of how long it would take to sweeten a cup of coffee if you just deposited a lump of sugar in it, and did not stir the coffee.

THE END.

[1]Generally ν is not quite constant, since is depends (for example) on the temperature.

[2]Carefully, so as not to start motion. The ink density must match that of water, to avoid gravity induced motion.

[3]Larger molecules diffuse slower than smaller ones; thus $\gamma(\text{sugar}) < \gamma(\text{salt})$. Also, γ grows with temperature, and may even depend on the concentration.