Name _____

18.311, Principles of Applied Mathematics, Spring 2005, Bazant

Midterm Exam – Thursday, March 31, 2005

Instructions: Please write your name on every page. This closed-book exam will last 90 minutes. Point totals for each problem are given out of 100.

1. (50 POINTS TOTAL) Consider the traffic-flow PDE,

$$\frac{\partial \rho}{\partial t} + u_m \left(1 - \frac{2\rho}{\rho_j}\right) \frac{\partial \rho}{\partial x} = 0. \tag{1}$$

with the initial traffic density,

$$\rho(x,0) = \begin{cases} \frac{\rho_j}{2} & \text{if } x \le 0 \\ \frac{\rho_j}{2}\left(1 + \frac{x}{d}\right) & \text{if } 0 < x < d \\ \rho_j & \text{if } x \ge d \end{cases} \tag{2}$$

(a) (25 POINTS) Solve for $t > 0$ until a shock forms, and sketch the time evolution of $\rho(x,t)$.

(b) (10 POINTS) Solve for $\rho(x,t)$ after the shock forms.

(c) (15 POINTS) Carefully draw a space-time diagram showing characteristics leaving the x axis for $t > 0$. Label the trajectories of weak discontinuities (starting at $x = 0$ and $x = d$), regions of constant density ($\rho = \rho_j/2$ and $\rho = \rho_j$), and the shock locus.

2. (25 POINTS) Solve the first-order quasi-linear PDE,

$$\psi_t + \psi\psi_x = 1$$

for $t > 0$ subject to the initial condition $\psi(x, 0) = x$.

3. (25 POINTS) Solve the linear wave equation,

$$u_{tt} = c^2 u_{xx},$$

for $x > 0$ and $t > 0$ for a semi-infinite string initially at rest, $u(x,0) = u_t(x,0) = 0$, displaced at one end according to $u(0,t) = f(t)$.