18.311 Principles of Applied Mathematics, Spring 2006, Prof. Bazant

Final Exam 2006

Instructions: Please begin the solution for each problem on a separate page. This is a three-hour, closed-book exam.

1. (15 points) The ice density $A(x,t)$ of a glacier (mass per cross-sectional area) is governed by the PDE,

$$A_t + \alpha A^{1/4} A_x = 0$$

   (a) Use dimensional analysis to estimate the time, t_s, for a shock (wall) to form in a region where the area drops from A_0 over a distance, λ. (What are the units of α?)

   (b) Find an implicit solution for $A(x,t)$ for early times (prior to shock formation) for the initial condition,

$$A(x,0) = f(x) = \begin{cases} A_0 & \text{for } x < 0 \\ A_0\left(1 - \frac{x}{2\lambda}\right) & \text{for } 0 < x < \lambda \\ \frac{A_0}{2} & x > \lambda \end{cases}$$

2. (15 points) Consider again the glacier dynamics in problem 1b, which leads to a single shock.

   (a) What is the limiting shock velocity as $t \to \infty$?

   (b) When and where does the shock form? Sketch the characteristic diagram.

3. (15 points) Solve for the traffic density with cars leaving the road (at a rate α),

$$\rho_t + c(\rho)\rho_x = -\alpha\rho$$

   after a red light turns green,

$$\rho(x,0) = \begin{cases} \rho_j & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

   for linear traffic flow, $c(\rho) = u_m(1 - 2\rho/\rho_j)$. Sketch the characteristic diagram, and plot the solution at several times.

4. (10 points) Consider the PDE

$$\rho_t + c_0\rho_x = \epsilon\rho_{xxx}$$

   (a) Find the dispersion relation.

   (b) With $\rho(x,0) = f(x)$, solve for $\rho(x,t)$ in integral form in terms of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

5. (10 points) Find a traveling wave solution to the diffusion equation

$$\rho_t = D\rho_{xx}$$

of the form $\rho(x,t) = \rho_o F(x - vt)$ with $F(0) = 0$ and $F(\infty) = 1$, which describes an absorbing interface at $x = vt$ invading a region of uniform concentration.

6. (10 points) Consider the intial-boundary-value problem:

$$\rho_t = a\,x\,\rho_{xx}, \quad x > 0, t > 0$$

$$\rho(x,0) = \rho_0, \quad \rho(0,t) = 0.$$

   (a) Use dimensional analysis to show that $\rho(x,t)$ has the form of a similarity solution. (What are the units of a?)

   (b) Solve for $\rho(x,t)$.

7. (10 points) The height $h(x,t)$ and depth-averaged fluid velocity $u(x,t)$ in shallow water satisfy

$$\begin{aligned} h_t + (uh)_x &= 0 \\ u_t + (\frac{1}{2}u^2 + gh)_x &= 0 \end{aligned}$$

where each PDE is a conservation law of the form $\rho_t + q_x = 0$. Consider a wall at $x = U_w t$ advancing for $t > 0$ into shallow water initially at rest, $h(x,0) = h_0$, $u(x,0) = 0$ for $x > 0$. This produces a shock wave ahead of the wall at $x = U_s t$ with $u(x,t) = U_w$ behind the shock.

   (a) Write down two expressions for the shock velocity, which determine U_s and h_w, the water height behind the shock. (Do not solve!)

   (b) Solve for h_w and U_s if $h_0 = g = 1$ and $U_w = \sqrt{2}$ (for a suitable choice of units).

8. (15 points) The electrostatic potential $\Phi(X)$ near an electrode at $X = 0$ applying a voltage V satisfies the Poisson-Boltzmann equation,

$$\varepsilon \frac{d^2\Phi}{dX^2} = eC \sinh\left(\frac{e\Phi}{kT}\right), \quad \text{with } \Phi(0) = V \text{ and } \Phi \to 0 \text{ as } x \to \infty,$$

where e is the ionic charge, C is the concentration of ions, ε is the permittivity, and kT/e is the thermal voltage.

   (a) Choose a length scale L to obtain the dimensionless problem,

$$\phi'' = \sinh(\phi), \quad \phi(0) = v, \phi(\infty) = 0$$

   where $\phi = e\Phi/kT$, $x = X/L$, and $v = eV/kT$.

   (b) Linearize for $|v| \ll 1$, and integrate twice to obtain an approximation for $\phi(x)$.

   (c) Without linearizing, integrate the ODE once to obtain $\phi'(x)$.