Final Exam 2006

Instructions: Please begin the solution for each problem on a separate page. This is a three-hour, closed-book exam.

1. (15 points) The ice density $A(x,t)$ of a glacier (mass per cross-sectional area) is governed by the PDE,

$$A_t + \alpha A^{1/4} A_x = 0$$

(a) Use dimensional analysis to estimate the time, $t_s$, for a shock (wall) to form in a region where the area drops from $A_0$ over a distance, $\lambda$. (What are the units of $\alpha$?)

(b) Find an implicit solution for $A(x,t)$ for early times (prior to shock formation) for the initial condition,

$$A(x,0) = f(x) = \begin{cases} A_0 & \text{for } x < 0 \\ A_0 \left(1 - \frac{x}{\lambda}\right) & \text{for } 0 < x < \lambda \\ \frac{A_0}{2} & \text{for } x > \lambda \end{cases}$$

2. (15 points) Consider again the glacier dynamics in problem 1b, which leads to a single shock.

(a) What is the limiting shock velocity as $t \to \infty$?

(b) When and where does the shock form? Sketch the characteristic diagram.

3. (15 points) Solve for the traffic density with cars leaving the road (at a rate $\alpha$),

$$\rho_t + c(\rho)\rho_x = -\alpha \rho$$

after a red light turns green,

$$\rho(x,0) = \begin{cases} \rho_j & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

for linear traffic flow, $c(\rho) = u_m(1 - 2\rho/\rho_j)$. Sketch the characteristic diagram, and plot the solution at several times.

4. (10 points) Consider the PDE

$$\rho_t + c_0 \rho_x = \epsilon \rho_{xxx}$$

(a) Find the dispersion relation.

(b) With $\rho(x,0) = f(x)$, solve for $\rho(x,t)$ in integral form in terms of the Fourier transform,

$$\hat{\rho}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$
5. (10 points) Find a traveling wave solution to the diffusion equation

\[
\rho_t = D\rho_{xx}
\]

of the form \( \rho(x, t) = \rho_0 F(x - vt) \) with \( F(0) = 0 \) and \( F(\infty) = 1 \), which describes an absorbing interface at \( x = vt \) invading a region of uniform concentration.

6. (10 points) Consider the initial-boundary-value problem:

\[
\rho_t = a x \rho_{xx}, \quad x > 0, \quad t > 0
\]

\[
\rho(x, 0) = \rho_0, \quad \rho(0, t) = 0.
\]

(a) Use dimensional analysis to show that \( \rho(x, t) \) has the form of a similarity solution. (What are the units of \( a \)?)

(b) Solve for \( \rho(x, t) \).

7. (10 points) The height \( h(x, t) \) and depth-averaged fluid velocity \( u(x, t) \) in shallow water satisfy

\[
\begin{align*}
h_t + (uh)_x &= 0 \\
u_t + \left( \frac{1}{2} u^2 + gh \right)_x &= 0
\end{align*}
\]

where each PDE is a conservation law of the form \( \rho_t + q_x = 0 \). Consider a wall at \( x = U_w t \) advancing for \( t > 0 \) into shallow water initially at rest, \( h(x, 0) = h_0, \ u(x, 0) = 0 \) for \( x > 0 \). This produces a shock wave ahead of the wall at \( x = U_s t \) with \( u(x, t) = U_w \) behind the shock.

(a) Write down two expressions for the shock velocity, which determine \( U_s \) and \( h_w \), the water height behind the shock. (Do not solve!)

(b) Solve for \( h_w \) and \( U_s \) if \( h_0 = g = 1 \) and \( U_w = \sqrt{2} \) (for a suitable choice of units).

8. (15 points) The electrostatic potential \( \Phi(X) \) near an electrode at \( X = 0 \) applying a voltage \( V \) satisfies the Poisson-Boltzmann equation,

\[
\varepsilon \frac{d^2\Phi}{dX^2} = eC \sinh \left( \frac{e\Phi}{kT} \right), \quad \text{with} \ \Phi(0) = V \ \text{and} \ \Phi \rightarrow 0 \ \text{as} \ x \rightarrow \infty,
\]

where \( e \) is the ionic charge, \( C \) is the concentration of ions, \( \varepsilon \) is the permittivity, and \( kT/e \) is the thermal voltage.

(a) Choose a length scale \( L \) to obtain the dimensionless problem,

\[
\phi'' = \sinh(\phi), \quad \phi(0) = v, \ \phi(\infty) = 0
\]

where \( \phi = e\Phi/kT \), \( x = X/L \), and \( v = eV/kT \).

(b) Linearize for \( |v| \ll 1 \), and integrate twice to obtain an approximation for \( \phi(x) \).

(c) Without linearizing, integrate the ODE once to obtain \( \phi'(x) \).