Name ______________________________

18.311, Principles of Applied Mathematics, Spring 2004, Bazant

Final Exam – Wednesday, May 19, 2004

Instructions: Please write your name on every page. This *closed-book* exam will last three hours. Point totals for each problem are given out of 100. Graded exams and solutions will be available starting tomorrow. I hope you enjoyed the class. – MZB

1. (10 POINTS) Consider a steady initial flow of traffic through a green light at uniform density, $\rho(x, 0) = \rho_0$, where $0 < \rho_0 < \rho_j$. Assuming $u(\rho) = u_{max}(1 - \rho/\rho_j)$, solve the PDE

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

with boundary conditions, $\rho(0^-, t) = \rho_j$ and $\rho(0^+, t) = 0$ for $t > 0$ after the light turns red.

2. (25 POINTS) Consider again traffic flow with $u(\rho) = u_{max}(1-\rho/\rho_j)$ for the initial condition,

$$\rho(x,0) = \frac{\rho_j/2}{1+|x|/d}$$

(a) Solve for $\rho(x,t)$ for $0 < t < t_s$, where t_s is the shock formation time. (You may leave your solution in implicit form.)

(b) Find the time, t_s, and the position, x_s, where a shock forms.

(c) Draw a characteristic diagram, giving a rough sketch of the shock locus (without solving for it).

(d) Sketch the solution, $\rho(x,t)$, at various times before and after shock formation. (Try this even if you did not solve the other parts.)

3. (15 POINTS) Use a Fourier transform (in x) to solve the linear reaction-diffusion equation:

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} - r\,\rho$$

subject to the initial condition, $\rho(x, 0) = \delta(x)$.

4. (10 POINTS) Find the dispersion relation, $\omega(k)$, for the forced wave equation,

$$\frac{\partial^2 \rho}{\partial t^2} = c_0^2 \frac{\partial^2 \rho}{\partial x^2} - \omega_0^2\, \rho,$$

and show that $u_p u_g = c_0^2$, where $u_p(k)$ is the phase velocity and $u_g(k)$ is the group velocity.

5. (15 POINTS) Consider the following "spreading" problem with space-dependent diffusion:

$$\frac{\partial c}{\partial t} = a \frac{\partial}{\partial x}\left(|x|^{\epsilon} \frac{\partial c}{\partial x} \right), \quad c(x, 0) = Q_0\, \delta(x).$$

(a) Use dimensional analysis to show that $c(x.t)$ has the form of a similarity solution,

$$c(x, t) = \frac{A}{(Bt)^{\alpha}}\, \Phi\left(\frac{x}{(Bt)^{\alpha}} \right).$$

What are the parameters, A, B, and α?

(b) Substitute the similarity form into the PDE and derive the scaling function, $\Phi(z)$. [Hint: since the solution must be even, $c(-x,t) = c(x,t)$, only solve for $x > 0$.]

6. (10 POINTS) Consider a very viscous, Newtonian fluid, whose velocity, $\vec{u}$, and pressure, P, satisfy the Stokes equations,

$$\nabla P = \eta \nabla^2 \vec{u}, \qquad \nabla \cdot \vec{u} = 0$$

which contains a spherical bubble of radius, R, and surface tension, γ. (Recall that P has units of energy/volume.)

(a) Suppose that the bubble is filled with a uniform gas at pressure, P_0, and remains in equilibrium. From the energy balance, $\gamma A = P_0 V$, where A and V are the surface area and volume of the bubble, respectively, calculate the "Laplace pressure", $P_0(\gamma, R)$.

(b) Now suppose that the bubble is empty, so that surface tension causes it to collapse in finite time, hindered only by the viscosity of the surrounding fluid. Use dimensional analysis to show how the bubble lifetime, t_c, depends on R, η, and γ.

7. (15 POINTS) Use the Method of Characteristics to solve

$$\rho \frac{\partial \rho}{\partial x} + 2e^x \frac{\partial \rho}{\partial y} = -\rho^2$$

subject to the boundary condition, $\rho = 1$ on the curve $y = e^x$.