Problem Set Number 04, 18.306
MIT (Winter-Spring 2021)

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Last day of Lectures, Spring 2021.
Turn it in via the canvas site website.

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1 Fundamental solution for the Laplace operator

Statement: Fundamental solution for the Laplace operator

Let Δ be the Laplace operator in $\mathcal{R}^d$

$$\Delta = \sum_{n=1}^{d} \partial_{x_n}^2, \tag{1.1}$$

where $d \geq 3$. Further, let G be defined by

$$G = \frac{1}{(d-2)\,\alpha_d\, r^{d-2}}, \quad \text{where} \quad r = \sqrt{\sum_{n=1}^{d} x_n^2} \tag{1.2}$$

and $\boldsymbol{\alpha_d}$ is **the "area" of the unit sphere $\boldsymbol{S^d(1)}$ in $\boldsymbol{d}$-dimensions. Show that**

$$\boldsymbol{\Delta G = -\delta(\vec{x})}, \tag{1.3}$$

where δ is Dirac's delta in $\mathcal{R}^d$. That is: show that *for any test function*[1] ψ *in* $\mathcal{R}^d$

$$\psi(0) + \int_{\mathcal{R}^d} G\,\Delta\psi\, dV = 0, \tag{1.4}$$

where $dV = dx_1 \ldots dx_d$ is the "volume" element in $\mathcal{R}^d$. Proceed as follows

1. For any function f in R^d, define the radial function $\overline{f} = \overline{f}(r)$ by averaging over the angular variables:

$$\overline{f} = \frac{1}{\alpha_d} \int_{S^d(1)} f(r, \vec{\sigma})\, dS = \frac{1}{\alpha_d\, r^{d-1}} \int_{S^d(r)} f\, d\tilde{S}, \tag{1.5}$$

where: (i) $\vec{\sigma}$ are the angular variables in R^d; (ii) dS is the surface element in $S^d(1)$ — dS involves only the angular variables; and (iii) $d\tilde{S} = r^{d-1}\, dS$ is the area element in $S^d(r)$ (sphere of radius r).

2. Use Gauss theorem to express $\overline{\psi}_r$ in terms of the integral of $\Delta\psi$ over a ball of radius r. That is, in terms of:

$$\int_{B^d(r)} \Delta\psi\, dV, \quad \text{where } B^d(r) = \{\vec{x} \in R^d \mid \|\vec{x}\| \leq r\}.$$

[1] A smooth function decaying as fast as you need as $r \to \infty$.

3. Use **2** to show that $$\Delta\overline{\psi} = \overline{\Delta\psi}. \tag{1.6}$$
Note: $\Delta g = r^{1-d}\,(r^{d-1}\,g_r)_r$ for radial functions.

4. Use the fact that $dV = r^{d-1}\,dr\,dS$ to re-write the integral in (1.4). You are now a small step away from being able to show (1.4) applies.

Important: You **do not need** explicit expressions for the angular variables.
This problem involves very few grungy calculation.

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2 Green's functions #11

Statement: Green's functions #11

The Green's function for the heat equation half line Dirichlet[2] signaling problem is defined by

$$G_t = G_{xx}, \quad \text{for} \quad -\infty < t < \infty \quad \text{and} \quad x > 0, \tag{2.1}$$

with the boundary condition $$G(0,\,t) = \delta(t), \tag{2.2}$$
where $\delta(\cdot)$ denotes Dirac's delta function. In addition: G is bounded away from the origin $(0,\,0)$ and satisfies **causality** $\boldsymbol{G = 0}$ **for** $\boldsymbol{t < 0}$.
Thus **we only need to find $\boldsymbol{G}$ for $\boldsymbol{t \geq 0}$ only.**

For any constant $\nu \neq 0$, $\boldsymbol{\nu^2\,\delta(\nu^2\,t) = \delta(t)}$. Thus: If G is a solution, $\nu^2\,G(\nu\,x,\,\nu^2\,t)$ is a solution.

Hence, from uniqueness, for any $\nu \neq 0$, $$G(x,\,t) = \nu^2\,G(\nu\,x,\,\nu^2\,t). \tag{2.3}$$

Set $\nu = x/t$ to get, for some function $g = g(\xi)$, $$G(x,\,t) = \frac{1}{t}\,g\left(\frac{x^2}{t}\right). \tag{2.4}$$

Notice that

A. $\boldsymbol{g(0) = 0}$. This follows because, for any $t > 0$ fixed, G must vanish as $x \downarrow 0$.

B. $\displaystyle\boldsymbol{\int_0^\infty g(\xi)\,\frac{d\xi}{\xi} = 1}$. We have $\displaystyle\int_{-\infty}^{\infty} G(x,\,t)\,dt = \int_0^\infty g(\xi)\,\frac{d\xi}{\xi} = \text{constant}$, for any $x > 0$.
By taking $x \downarrow 0$, and using (2.2), the result follows.

- **Task 1:** Substitute (2.4) into (2.1), and get an o.d.e. for g.
- **Task 2:** Solve the o.d.e. for g, and thus find the Green's function.

Hints: (i) *The o.d.e. for g is second order. It can be integrated once, to yield a first order equation.* **(ii)** *The general solution to the o.d.e. in* **1** *has two constants. These follow from* **A – B** *above.*

Remark 2.1 *For any fixed $x > 0$, the formula in (2.4) should, as $t \downarrow 0$, smoothly match with the solution $G \equiv 0$ for $t < 0$. This will be guaranteed by the fact that $g(\xi)$ vanishes exponentially as $\xi \to \infty$. Hence, G as given by (2.4), as well as all its derivatives, vanish as $t \downarrow 0$ for any $x > 0$.* ♣

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[2] Temperature prescribed on the boundary.

3 Laplace equation in a circle #01

Statement: Laplace equation in a circle #01

Green's function: Laplace equation in a circle, with Dirichlet BC. Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) are insulated. This problem can be written — using polar coordinates and non-dimensional variables (selected so that the plate has radius 1) — in the form

$$\frac{1}{r^2}\left(r\,(r\,T_r)_r + T_{\theta\theta}\right) = \Delta\,T = 0, \quad \text{for } 0 \le \theta \le 2\pi \quad \text{and } 0 \le r \le 1, \tag{3.1}$$

with the boundary condition † $\quad T(1,\,\theta) = h(\theta), \quad$ for some given function h. (3.2)

† T and h are $2\,\pi$-periodic in θ.

The solution to this problem, as follows from separation of variables, is [3]

$$T = \sum_{n=-\infty}^{\infty} h_n\, r^{|n|}\, e^{i\,n\,\theta}, \quad \text{where} \quad h_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} h(\theta)\, e^{-i\,n\,\theta} d\theta. \tag{3.3}$$

For $r < 1$ this formula provides a solution to the equation even if h is not very smooth (or in fact, not even a function) and thus has a very poorly convergent Fourier series $\sum h_n\, e^{i\,n\,\theta}$. For example, assume that the sequence $\{h_n\}$ is bounded. Then, in any disk of radius $R < 1$: (i) the series for T in (3.3) converges absolutely; and (ii) the series obtained by term by term differentiation, for any derivative of T, also converges absolutely. It follows that T is smooth for $r < 1$. In a similar fashion, one can justify exchanging the order of integration and summation — as you are asked to do below.

1. Assume that $h = \delta(\theta)$, and **write an explicit formula for the solution T.**
2. In (3.3), substitute the formula for h_n into the series for T, exchange the summation and integration order, and

   derive an equation of the form $\quad T = \int_0^{2\,\pi} G(r,\,\theta-\phi)\,h(\phi)\,d\phi.$ (3.4)
3. **Show that:**

   (a) $G > 0$ for $r < 1$. (b) $G(r,\,\theta) \to 0$ as $r \to 1$, if $\theta \ne 0$. (c) $\int_0^{2\,\pi} G(r,\,\theta)\,d\theta = 1$ for $r < 1$.

Hint. Add the positive and negative indexes n separately. Then use that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ *for* $|z| < 1$.

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4 Laplace equation in a circle #02

Statement: Laplace equation in a circle #02

Green's function: Laplace equation in a circle, with Neumann BC. Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature flux is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) are insulated. This problem can be written — using polar coordinates and non-dimensional variables (selected so that the plate has radius 1) — in the form

$$\frac{1}{r^2}\left(r\,(r\,T_r)_r + T_{\theta\theta}\right) = \Delta\,T = 0, \quad \text{for } 0 \le \theta \le 2\pi \quad \text{and } 0 \le r \le 1, \tag{4.1}$$

[3] Check, by direct substitution, that this is indeed the solution. That (3.2) is satisfied should be "obvious".

with the boundary condition † $\quad T_r(1,\,\theta) = h(\theta), \quad$ for some given function h. (4.2)

† T and h are 2π-periodic in θ.

In addition, the function $h = h(\theta)$ must satisfy
$$\int_0^{2\pi} h(\theta)\,d\theta = 0. \tag{4.3}$$
Equation (4.3) follows because, unless the net heat flux through the boundary vanishes, no steady state temperature is possible. Equivalently: *(4.1–4.2) does not have a solution unless (4.3) applies.*

The solution to this problem, as follows from separation of variables, is [4]
$$T = \sum_{-\infty<n<\infty \quad n\neq 0} \frac{1}{|n|}\, h_n\, r^{|n|}\, e^{i\,n\,\theta}, \qquad \text{where} \quad h_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} h(\theta)\, e^{-i\,n\,\theta} d\theta, \tag{4.4}$$
where, **to make the solution unique,**[5] **we have added the condition: $T = 0$ at the origin.**

For $r < 1$ this formula provides a solution to the equation even if h is not very smooth (or in fact, not even a function) and thus has a very poorly convergent Fourier series $\sum h_n\, e^{i\,n\,\theta}$. For example, assume that the sequence $\{h_n\}$ is bounded. Then, in any disk of radius $R < 1$: (i) the series for T in (4.4) converges absolutely; and (ii) the series obtained by term by term differentiation, for any derivative of T, also converges absolutely. It follows that T is smooth for $r < 1$. In a similar fashion, one can justify exchanging the order of integration and summation — as you are asked to do below.

1. Assume that $h = \delta(\theta) - \dfrac{1}{2\,\pi}$, and **write an explicit formula for the solution T.**
2. In (4.4), substitute the formula for h_n into the series for T, exchange the summation and integration order, and

   derive an equation of the form
$$T = \int_0^{2\,\pi} G(r,\,\theta-\phi)\, h(\phi)\, d\phi. \tag{4.5}$$
3. Let $H = G_r + \dfrac{1}{2\,\pi}$. **Show that:**

   (a) $H(r,\,\theta) \to 0$ as $r \to 1$, if $\theta \neq 0$. (b) $\displaystyle\int_0^{2\,\pi} H(r,\,\theta)\, d\theta = 1$ for $r < 1$.

Hint. Add the positive and negative indexes n separately. Then use $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n}\, z^n = -\ln(1-z)}$ *for* $|z| < 1$ *— note that, for* $|z| < 1$ *the argument for the log stays within a circle of radius 1 centered at 1, so that no problem arises because of the branch point of the log at the origin.*

5 Normal modes #02

Statement: Normal modes #02

Normal mode expansions are not guaranteed to work when the associated eigenvalue problem is not self adjoint (more generally, not normal).[6] Here is an example: consider the problem
$$u_t + u_x = 0 \quad \text{for} \;\; 0 < x < 1 \quad \text{and} \quad t > 0, \qquad \text{with} \;\; u(x,\,0) = U(x) \tag{5.1}$$
and boundary condition $\boldsymbol{u(0,\,t) = 0}$. **Find all the normal mode solutions** to this problem,[7] and **show that the solution to (5.1) cannot be written as a linear combination of these modes. What is the solution to (5.1)?**

THE END

[4] Check, by direct substitution, that this is indeed the solution. That (4.2) is satisfied should be "obvious".
[5] A constant can be added to any solution of (4.1–4.2).
[6] An operator A is normal if it commutes with its adjoint: $A\,A^\dagger = A^\dagger\,A$.
[7] That is: non-trivial solutions of the form $u = e^{\lambda\,t}\,\phi(x)$, where λ is a constant.