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Due Wed May 5, Spring 2021.
Turn it in via the canvas site website.

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1 Information loss and traffic flow shocks

Statement: Information loss and traffic flow shocks

In this problem we will show that shocks in traffic flow are associated with information loss, and will derive a formula that quantifies this loss. For simplicity, consider periodic solutions to the traffic flow equation

$$\rho_t + q_x = 0, \quad \text{periodic of period } T > 0, \tag{1.1}$$

where the traffic flux is given by $q = Q(\rho)$. Assume that there is a single shock per period, at $x = \sigma(t)$. The shock velocity, s, is given by the Rankine-Hugoniot jump condition

$$\frac{d\sigma}{dt} = s = \frac{q_R - q_L}{\rho_R - \rho_L}, \tag{1.2}$$

where ρ_R (resp.: ρ_L) is the value of ρ immediately to the right (resp.: the left) of the shock discontinuity, $q_R = Q(\rho_R)$, and $q_L = Q(\rho_L)$. In addition, the shock satisfies the *entropy condition*

$$c_L > s > c_R, \tag{1.3}$$

where $c = \dfrac{dQ}{d\rho}$ is the characteristic speed, $c_R = c(\rho_R)$, and $c_L = c(\rho_L)$. Now, let

$$\mathcal{I} = \mathcal{I}(t) = \int_{\text{period}} \frac{1}{2}\,\rho^2(x,\,t)\,dx. \tag{1.4}$$

1. **Show that:**
$$\frac{d\mathcal{I}}{dt} = \frac{1}{2}\,(q_L - q_R)\,(\rho_L + \rho_R) + \int_{\rho_L}^{\rho_R} \rho\, c(\rho)\, d\rho. \tag{1.5}$$

   Hint. For any short enough time period, a constant "a" exists such that $a < \sigma(t) < a + T$ during the time interval. Given this, write $\mathcal{I}$ in the form $\boldsymbol{\mathcal{I} = \int_a^\sigma \frac{1}{2}\,\rho^2\,dx + \int_\sigma^{a+T} \frac{1}{2}\,\rho^2\,dx}$, before taking the time derivative. Then use the fact that, in each of the two intervals ρ satisfies
$$\rho_t = -c(\rho)\,\rho_x = -\frac{1}{\rho}\,f_x \quad \textit{where} \quad f = \int_0^\rho s\,c(s)\,ds. \tag{1.6}$$
   In addition, use (1.2) to eliminate $\boldsymbol{\dot\sigma}$ from the resulting formulas.

2. **Show that:** $$\frac{d\mathcal{I}}{dt} < 0 \tag{1.7}$$
by interpreting the right hand side in (1.5) geometrically, as (the negative of) the area between two curves — such area known to be positive. *Hint. Use that* $\boldsymbol{\rho\, c = \frac{d}{d\rho}(\rho\, q) - q}$, *and (1.9) below.*

Remark 1.1 *Recall that, in traffic flow,* $q = Q(\rho)$ *is a concave function,*[1] *vanishing at both* $\rho = 0$ *and the jamming density* $\rho_j > 0$*, positive for* $0 < \rho < \rho_j$*, with a maximum (the road capacity* q_m*) at some value* $0 < \rho_m < \rho_j$*. Because of this,* **the conditions (1.2–1.3) are equivalent to:**[2]

$$\textbf{The shock velocity is given by the slope of the secant line joining } (\rho_L,\, q_L) \textbf{ to } (\rho_R,\, q_R)\textbf{. Further } \rho_L < \rho_R. \tag{1.8}$$

Note that **For $\rho_L < \rho < \rho_R$, the curve $q = Q(\rho)$ is *above* the secant line,** *(1.9)*
because Q is concave.

Remark 1.2 *How does $\mathcal{I}$ in (1.4) related to "information"? Basically, we argue that a function contains the least amount of information when it is a constant, and that the more "oscillations" it has, the more information it carries. A (rough) measure of this is given by*

$$\mathcal{J} = \int_{\text{period}} \frac{1}{2}\,(\rho - \overline{\rho})^2\, dx, \tag{1.10}$$

where $\overline{\rho}$ is the average value of ρ (note that conservation guarantees that $\overline{\rho}$ is a constant in time). Alternatively, write ρ the Fourier series $\rho = \sum_n \rho_n(t)\, e^{i\, n\, 2\, \pi\, x/T}$ for ρ. Then the amount of "oscillation" in ρ can be characterized by $\sum_{n\neq 0} \frac{1}{2}\,|\rho_n|^2$, which is the same as (1.10). It is easy to see that

$$\mathcal{J} = \mathcal{I} - T\,\overline{\rho}^2. \tag{1.11}$$

Hence $\dot{\mathcal{J}} = \dot{\mathcal{I}}$, so that (1.7) indicates that the shock is driving ρ towards a constant value.

The notion of "information" that we use here is not the same as that in "information theory" (which requires a stochastic process of some kind). Neither is it the same as the related notion of "entropy" of statistical mechanics (for this we would need a statistical theory[3] *for traffic flow). Nevertheless, $S = -\mathcal{J}$ plays the same role for (1.1) that entropy plays for gas dynamics.*

Finally, we point out that it can be shown that: for any convex function $h = h(\rho)$ (i.e.: $h'' > 0$)

$$\frac{d}{dt}\int_{\text{period}} h(\rho)\, dx < 0 \quad \textit{when shocks are present,}$$

though showing this is a bit more complicated than for the special case $h = \frac{1}{2}\,\rho^2$.

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2 IVP for a kinematic wave equation #01

Statement: IVP for a kinematic wave equation #01

Consider the Kinematic Wave equation

$$u_t + q_x = 0, \qquad \text{where} \qquad q = \frac{1}{2}\,u^2, \tag{2.1}$$

[1] Concave means that $\frac{d^2 Q}{d\rho^2} < 0$. Note also that **only solutions satisfying $0 \le \rho \le \rho_j$ are acceptable.**
[2] Thus shocks move slower than cars, and the density a driver sees increases as the car goes through a shock.
[3] Such theories exist.

and u is the density for some conserved quantity. Using the method of characteristics, the Rankine-Hugoniot jump conditions, and the entropy conditions, find the solutions (for **all** $t > 0$) to the following initial value problems:

1st Initial Value Problem:

$$u(x,\,0) = \begin{cases} 1 & \text{for} \quad -\infty < x \leq -1, \\ -x & \text{for} \quad -1 \leq x \leq 0, \\ 0 & \text{for} \quad 0 \leq x < \infty. \end{cases} \tag{2.2}$$

2nd Initial Value Problem:

$$u(x,\,0) = \begin{cases} 0 & \text{for} \quad -\infty < x \leq -1, \\ 1+x & \text{for} \quad -1 \leq x \leq 0, \\ 1-x & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{for} \quad 1 \leq x < \infty. \end{cases} \tag{2.3}$$

3 Smooth kinematic waves have infinite conservation laws

Statement: Smooth kinematic waves have infinite conservation laws

Consider the (kinematic wave) equation for some conserved scalar density $\rho = \rho(x,\,t)$ in one dimension

$$\rho_t + q_x = 0, \quad \text{where} \quad q = Q(\rho) \tag{3.1}$$

is the flow function — which we assume is sufficiently nice (say, it has a continuous first derivative). **SHOW THAT:** if $f = f(\rho)$ is some arbitrary (sufficiently nice) function, then there exists a function $g = g(\rho)$ such that *for any classical*[4] *solution of (3.1), we have*

$$f_t + g_x = 0. \tag{3.2}$$

In other words, (formally) f behaves as a "conserved" quantity, with flow function g. In particular:

$$\frac{d}{dt}\int_a^b f\,dx = g\big|_{x=a} - g\big|_{x=b} \tag{3.3}$$

expresses the "conservation" of f for any interval $a < x < b$.

Very important: it is *crucial that the solution ρ have derivatives in the classical sense.* **When shocks are present, this result is false.** As you will be asked to show in another problem, when shocks are present (3.2) has to be replaced by

$$f_t + g_x = \sum_{\text{shocks}} c_s\,\delta(x - x_s(t)), \tag{3.4}$$

where $x = x_s(t)$ are the shock positions, δ is the Dirac delta function, and the $c_s = c_s(t)$ are some coefficients that are *generally not zero.* Thus the shocks act as sources (or sinks) for f.

[4] That is: no shocks, so no generalized derivatives are involved.

4 The zero viscosity limit for scalar convex conservation laws

Statement: The zero viscosity limit for scalar convex conservation laws

Consider a scalar convex conservation law, with a small amount of "viscosity" added. Namely:

$$\rho_t + q_x = \nu\,\rho_{x\,x}, \quad \text{with } \ q = q(\rho) \ \text{ smooth and convex: } \frac{d^2 q}{d\rho^2} \geq C_q > 0, \ \text{ and } \ \rho(x,\,0) = f(x), \tag{4.1}$$

where C_q is some constant, $0 < \nu \ll 1$, f is smooth, and f and all its derivatives vanish (rapidly) as $|x| \to \infty$. We want to investigate the behavior, as $\nu \to 0$, of the solution to this problem. In particular, **we want to compute**

$$\mathcal{I} = \mathcal{I}(t) = \int_{-\infty}^{\infty} \Psi(\rho(x,\,t))\,dx, \quad \text{as } \ \nu \to 0, \tag{4.2}$$

where Ψ is a smooth convex function [5] such that $\Psi(0) = 0$. Furthermore, let $h = h(\rho)$ be defined by

$$\frac{dh}{d\rho} = c(\rho)\,\frac{d\Psi}{d\rho} \ \text{ and } \ h(0) = 0, \quad \text{where} \quad c = \frac{dq}{d\rho}. \tag{4.3}$$

Note 1. We will assume that the solution $\rho = \rho(x,\,t)$ to (4.1) exists, that it is smooth, that ρ and all its derivatives vanish (rapidly) as $|x| \to \infty$, and that ρ is unique (within this class).

Note 2. Here we will use concepts introduced in the problem Lax Entropy condition for scalar convex conservation laws and information loss — you should read the statement for this exercise. This will also serve the purpose of giving meaning to $\mathcal{I}$ as a measure of the "wavy-ness" of the solution.

Multiplying by $\dfrac{d\Psi}{d\rho}$ the equation $\rho_t + c(\rho)\,\rho_x = \nu\,\rho_{x\,x}$ satisfied by ρ, and using (4.3), we obtain

$$\Psi(\rho)_t + h(\rho)_x = \nu\,\Psi'(\rho)\,\rho_{x\,x}, \tag{4.4}$$

where the prime denotes differentiation with respect to ρ. Integrating this equation then yields

$$\frac{d\mathcal{I}}{dt} = \nu \int_{-\infty}^{\infty} \Psi'(\rho)\,\rho_{x\,x}\,dx = -\nu \int_{-\infty}^{\infty} \Psi''(\rho)\,\rho_x^2\,dx < 0, \tag{4.5}$$

which shows that $\mathcal{I}$ is, always, a decreasing function of time.

From (4.5) is seems natural to conclude that: in the limit $\nu \to 0$, $\mathcal{I}$ becomes constant. However, **this is false.** As $\nu \to 0$, the solution to (4.1) develops thin transition layers (shocks) where the derivatives become large, so that the right hand side in (4.5) does not vanish as $\nu \to 0$. **Your task here is to check this fact.** *Proceed as follows.*

First, consider the case where the solution to (4.1) develops a single shock as $\nu \to 0$, along some path $x = \sigma(t)$. Then,

For $0 < \nu \ll 1$, the solution near $x = \sigma$ (for any fixed time t) can be described by a traveling wave solution to the equation in (4.1), of the form

$$\rho \sim \mathcal{R}(z), \quad \text{where} \quad z = \frac{x - s\,t}{\nu} \quad \text{and} \quad s = \frac{d\,\sigma}{d\,t}, \tag{4.6}$$

with limits ρ_R as $z \to \infty$ and ρ_L as $z \to -\infty$. On the other hand, away from $x = \sigma$, the derivatives of ρ remain bounded as $\nu \to 0$.

Use (4.6) and (4.5) to **compute the $\nu \to 0$ limit** of $\dfrac{d\mathcal{I}}{dt}$. You should be able to **show that**

$$\frac{d\mathcal{I}}{dt} \quad \longrightarrow \quad h_R - h_L - s\,(\Psi_R - \Psi_L), \tag{4.7}$$

[5] Hence $\dfrac{d^2\,\Psi}{d\rho^2} \geq C_\Psi > 0$, for some constant C_Ψ.

where the subscript R indicates evaluation at ρ_R, and the subscript L indicates evaluation at ρ_L. *Notice that this is the same formula for the contribution to the rate of change of $\mathcal{I}$ by a shock* listed in the statement for the problem Lax Entropy condition for scalar convex conservation laws and information loss — see the 9^{th} equation there.

Hint. Consider the first integral in (4.5). Away from $x = \sigma$, the derivatives of the solution remain bounded, and the contribution of these regions to the integral vanishes as $\nu \to 0$. Hence we can limit the integration to a small region near the shock, namely

$$\frac{d\mathcal{I}}{dt} \approx \nu \int_{\sigma-\epsilon}^{\sigma+\epsilon} \Psi'(\rho)\, \rho_{x\,x}\, dx, \quad 0 < \nu \ll \epsilon \ll 1, \tag{4.8}$$

where the traveling wave approximation $\rho \sim \mathcal{R}(z)$ in (4.6) can be used. From this, using the o.d.e. that $\mathcal{R}$ satisfies,[6] the result in (4.7) follows.

Second, if the solution to (4.1) develops more than one shock as $\nu \to 0$, then a calculation as the one above can be done near each of the shock locations $x = \sigma_\ell(t)$, obtaining as the result that the right hand side in (4.7) is replaced by a sum including the contributions for each one of the shocks.

Remark 4.1 *The result in this problem illustrates a persistent phenomena in nonlinear p.d.e. that incorporate "small" perturbations involving the highest derivatives in the equation. Then, as the perturbations vanish, their effects on the solution* **do not.** *For example: in high Reynolds number flows thin boundary layers can form (mostly near walls), where viscous effects dominate, and contribute a finite amount (that does not go away as the Reynolds number goes to infinity) to the flow behavior. In other cases high frequency (thin) structures appear over large regions of the solution, which again do not go away as the perturbations vanish. This second type of situation is quite often associated with open problems where a good mathematical theory is lacking — e.g.: collision-less shocks in plasmas, turbulence, etc.*

5 Traveling waves for the KdV-Burgers equation

Statement: Traveling waves for the KdV-Burgers equation

Consider the p.d.e. (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\, u_{x\,x} + \mu\, u_{x\,x\,x}, \quad \text{where } \nu > 0 \text{ and } \mu \neq 0 \text{ are both small.} \tag{5.1}$$

This p.d.e. has been proposed as a simple model for the structure of bores in shallow water.[7]

Your task is: Study the behavior of the physically meaningful traveling wave solutions of this equation, as a function of the equation parameters ν, μ, and any wave parameters (e.g.: wave amplitude, speed, etc.). Are there solutions that approach constants as $x \pm \infty$? If so, how are the constants approached (monotone, oscillations)? Are the constants equal, or different? Are there periodic, oscillatory solutions? Make plots sketching the types of solutions that are possible.

Note: to be **physically meaningful**, a solution has to be **bounded.**

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[6] Note that **you do not need to have an explicit formula for** $\mathcal{R}$. Knowing the o.d.e. that $\mathcal{R}$ satisfies, and knowing the limits as $z \to \pm\infty$ for $\mathcal{R}(z)$, **should be enough.**

[7] See §13.15 of Whitham's book: *Linear and nonlinear waves.*

HINT. Exploit problem symmetries to reduce the number of free parameters: Because the equation is Galilean invariant, you need only look at time independent solutions (zero propagation speed):

$$\left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{x\,x} + \mu\,u_{x\,x\,x}, \tag{5.2}$$

This can be integrated to

$$\mu\,u_{x\,x} = -\nu\,u_x + \frac{1}{2}\,u^2 + \kappa, \tag{5.3}$$

where κ is a constant. By a re-scaling of the dependent and independent variables, this last equation can be reduced to one of the following three cases, each having a single parameter

$$u_{x\,x} = -\delta\,u_x + \frac{1}{2}\,u^2 + \sigma, \quad \text{where } \sigma = \pm 1 \text{ or } \sigma = 0, \text{ and } 0 < \delta < \infty. \tag{5.4}$$

Study the physically relevant solutions for these equations, as δ varies.

Remark 5.1 *There is no explicit solution for (5.4). One approach is to write (5.4) as a phase plane system, and find the critical points and their type, the null-clines, etc. Then use this information to deduce the phase portrait for the equation. From this the answer to the problem should follow.*

THE END