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Due Friday April 9, Spring 2021.
Turn it in via the canvas site website.

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1 Compute a channel flow rate function #01

Statement: Compute a channel flow rate function #01

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{1.1}$$

where $A = A(x,t)$ is the cross-sectional filled area of the river bed, x measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate q should be a function of A only [1] follows from the assumption of quasi-equilibrium. Then q is determined by a local balance between the friction forces and the force of gravity down the river bed.

Assume now a man-made channel, with uniform triangular cross-section† and a uniform (small) downward slope, characterized by an angle θ. Assume also that the frictional forces are proportional to the product of the flow velocity

[1] Possibly also x. That is: $q = Q(x, A)$, to account for non-uniformities along the river.

u down the channel, and the wetted perimeter P_w of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function Q should have.** † Isosceles triangle, with bottom angle ϕ.

Hints: (1) $Q = u\,A$, *where u is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter P_w is proportional to some power of A.*

2 Gas flow through a porous media

Statement: Gas flow through a porous media

Consider the situation where a gas is flowing through a porous media. If conditions are not varying too rapidly, [2] the gas flow velocity through the media is determined by the balance of the pressure force driving the flow, and the viscous resistance to the flow through the pores. This balance is subsumed by *Darcy's law*

$$\vec{u} = -\frac{\kappa}{\mu}\,\nabla p, \tag{2.1}$$

where: $\vec{u}$ is the gas flow velocity, κ is the *permeability* of the porous media, μ is the *dynamic viscosity of the gas*, ∇ is the gradient operator, and p is the pressure. Units: $[\kappa] = \text{length}^2$, $[\mu] = \frac{\text{mass}}{\text{length}\times\text{time}}$, while $[p] = \frac{\text{force}}{\text{area}}$.

Consider now *a 1D flow of an ideal gas.*

— For example: flow down a pipe filled with sand, where the pipe has constant cross-section.

— For an ideal gas $p = R\,T\,\rho$, where R = gas constant, T = temperature, and ρ = gas density.

QUESTIONS:

1. Assume that *the flow occurs at constant temperature*, and **derive an equation for $\rho = \rho(x,\, t)$.**

2. If the flow is adiabatic, instead of isothermal, $p = \beta\,\rho^\gamma$, where β is a constant and γ is the ratio of specific heats for the gas. **What is the equation for $\rho = \rho(x,\, t)$ in this case?**

3 Linear 1st order PDE (problem 01)

Statement: Linear 1st order PDE (problem 01)

Part 1. Find the general solutions to the two 1st order linear scalar PDE

$$x\,u_x + y\,u_y = 0, \qquad \text{and} \qquad y\,v_x - x\,v_y = 0. \tag{3.1}$$

Hint: *The general solutions take a particular simple form in polar coordinates.*

Part 2. For u, find the solution such that on the circle $x^2 + y^2 = 2$, it satisfies $u = x$. *Where is this solution determined by the data given?*

Part 3. Is there a solution to the equation for v such that $v(x, 0) = x$, for $-\infty < x < \infty$?

Part 4. How does the general solution for u changes if the equation is modified to

$$x\,u_x + y\,u_y = (x^2 + y^2)\,\sin(x^2 + y^2)? \tag{3.2}$$

[2] The standard "quasi-equilibrium" assumption used when deriving continuum equations from conservation laws.

4 Linear 1st order PDE (problem 03)

Statement: Linear 1st order PDE (problem 03)

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{4.1}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to find the solution. Write, **explicitly**, $\boldsymbol{u = u(x,\,y)}$ as a function of x and y, using f.
Hint. Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$ (where $-\infty < \tau < \infty$). Finally: eliminate τ, to get u as a function of x and y.

Part 2. In which part of the $(x,\,y)$ plane is the solution determined?
Hint. Draw in the x-y plane the characteristics computed in part 1.

Part 3. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

5 Linear 1st order PDE (problem 10)

Statement: Linear 1st order PDE (problem 10)

Integrating factors. Show that the pde

$$(a(x,\,y)\,\mu)_y = (b(x,\,y)\,\mu)_x \tag{5.1}$$

is a necessary and sufficient condition guaranteeing that $\mu = \mu(x,\,y) \neq 0$ is an integrating factor for the ode

$$a(x,\,y)\,dx + b(x,\,y)\,dy = 0, \tag{5.2}$$

in any open subset of the plane without holes.

Part II. Assume that $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$.
Find an integrating factor for (5.2) — i.e.: obtain a nontrivial solution of (5.1). **Use it to integrate** (5.2), and **write (5.2) in the form**

$$\Phi(x,\,y) = \text{constant}, \quad \text{for some function } \Phi. \tag{5.3}$$

Hint 5.1 *Solving by characteristics (5.1) leads to (5.2), or equivalent (as part of the process — check this!) To* get out of this circular situation, note that: *for a and b as above, $\mu = F(x,\,y)$ solves (5.1) iff $\mu = F(y,\,x)$ does. This suggests that you should look for solutions*[3] *invariant under this symmetry; that is: $\mu(x,\,y) = \mu(y,\,x)$. Hence* write $\boldsymbol{\mu = \mu(u,\,v)}$, with $\boldsymbol{u = x + y}$ and $\boldsymbol{v = x\,y}$, *since solutions that satisfy $\mu(x,\,y) = \mu(y,\,x)$ must have this form — see remark **5.1**.*

Remark 5.1 *The transformation $(x,\,y) \to (u,\,v)$ is not one to one: it maps the whole xy-plane into the region $v \leq \frac{1}{4}\,u^2$ of the uv-plane, with double valued inverse $x = \frac{1}{2}\left(u \pm \sqrt{u^2 - 4\,v}\right)$ and $y = \frac{1}{2}\left(u \mp \sqrt{u^2 - 4\,v}\right)$. Furthermore:* **(a)** *The two inverses are related by the $x \leftrightarrow y$ switch.* **(b)** *The singular line $u^2 = 4\,v$ corresponds to the line $x = y$.* **(c)** *The map is a bijection between the regions $x \leq y$ and $4\,v \leq u^2$.* **(d)** *The map is a bijection between the regions $x \geq y$ and $4\,v \leq u^2$.*
From **(c - d)** *we see that: for* any *$\mu = \mu(x,\,y)$, $\mu = f(u,\,v)$ for $x \leq y$ and $\mu = g(u,\,v)$ for $x \geq y$, for some f and g. Then,* if $\boldsymbol{\mu(x,\,y) = \mu(y,\,x)}$, *$f = g$, so that* $\boldsymbol{\mu}$ has the form $\boldsymbol{\mu = \mu(u,\,v)}$.

[3] Note that you only need **one** nontrivial solution.

Part III. Why is it that this approach cannot be generalized to three variables? That is, to find integrating factors for equations of the form

$$a(x,\, y,\, z)\, dx + b(x,\, y,\, z)\, dy + c(x,\, y,\, z)\, dz = 0. \tag{5.4}$$

6 Semi-Linear 1st order PDE (problem 02)

Statement: Semi-Linear 1st order PDE (problem 02)

Discuss the problems

$$u_x + 2\,x\,u_y = 2\,x\,u^2, \qquad \text{with} \qquad \begin{cases} \text{(a) } u(x,\, x^2) = 1 & \text{for } -1 < x < 1, \\ \text{(b) } u(x,\, x^2) = -\pi/(1+\pi\,x^2) & \text{for } -1 < x < 1, \\ \text{(c) } u(x,\, x^2) = (1 - x^2/4)^{-1}/4 & \text{for } -1 < x < 1. \end{cases} \tag{6.1}$$

How many solutions exist in each case? Where are they uniquely defined?
Note that the data in these problems is prescribed along a characteristic!

7 Simple finite differences for a 1st order PDE

Statement: Simple finite differences for a 1st order PDE

Consider the linear PDE

$$u_t + (c(x)\,u)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4}\,\cos(x), \tag{7.1}$$

and $\boldsymbol{u}$ **is periodic of period** $\boldsymbol{2\,\pi}$ — i.e.: $\boldsymbol{u(x + 2\,\pi,\, t) = u(x,\, t)}$. Assume now that you are asked to calculate the solution of this p.d.e. for $\boldsymbol{0 \le t \le T = 6}$, with **initial condition** given by

$$u(x,\, 0) = u_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \le x \le \pi, \tag{7.2}$$

extended periodically outside the interval $[-\pi,\, \pi]$.

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:

$$x_n = -\pi + n\,h \quad \text{for} \quad 1 \le n \le N, \qquad \text{and} \qquad t_m = m\,k \quad \text{for} \quad 0 \le n \le M, \tag{7.3}$$

where N and M are "large" integers, $\boldsymbol{h = 2\,\pi/N}$, and $\boldsymbol{k = T/M}$. Let $\boldsymbol{u_n^m}$ **be the numerical solution's grid point values.** The expectation is that these values will be related to the exact solution $u(x,\, t)$ by $\boldsymbol{u_n^m = u(x_n,\, t_m) +}$ **small error**, with the error vanishing as N and M grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in h or k. **In all cases the formulas apply for** $\boldsymbol{m \ge 0}$**, with** $\boldsymbol{u_n^0 = u_0(x_n)}$**.**

A. $\boldsymbol{0 = \dfrac{1}{k}\,(u_n^{m+1} - u_n^m) + \dfrac{1}{h}\,\Big(c(x_n)\,u_n^m - c(x_{n-1})\,u_{n-1}^m\Big)}$.

B. $\boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_n)\,u_n^m\Big)}$.

C. $\boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{2\,h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_{n-1})\,u_{n-1}^m\Big)}$.

In all cases, once u_n^m is known for some m and all n, u_n^{m+1} can be explicitly computed, for all n. **Note:** when a formula above in **A**, **B** or **C**, calls for a value u_n^m outside the range $1 \le n \le N$, the periodic boundary conditions, which translate into $\boldsymbol{u_{n+N}^m = u_n^m}$, must be used.

The **tasks in this problem are:**

1. **Causality and numerics.** Using arguments based **solely** on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** *(1.1) One of the schemes above cannot possibly work. (1.2) A necessary condition for the other two to work is that a restriction of the form $k \le \lambda\, h$ be imposed on the time step* — where $\lambda > 0$ is a constant that depends on $c = c(x)$.
2. Implement the schemes and try them out with various space resolutions,[4] and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item **1**? **Report what you see, and illustrate it with plots** — a **few** well selected plots should be enough!

8 The flux for a conserved quantity must be a vector

Statement: The flux for a conserved quantity must be a vector

Note: Below (for simplicity) we present many arguments/questions in 2D. However they apply just as well in nD; $n = 3,\, 4,\, \ldots$

Consider some conserved quantity, with density $\rho = \rho(\vec{x}, t)$ and flux vector $\vec{q} = \vec{q}(\vec{x}, t)$ in 2D. Then, in the absence of sources or sinks, we made the argument that conservation leads to the (integral) equation

$$\frac{d}{dt}\int_{\Omega} \rho \,\mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\partial\Omega} \vec{q}\cdot\hat{n}\,\mathrm{d}s, \tag{8.1}$$

for any region Ω in the domain where the conserved "stuff" resides, where $\partial\Omega$ is the boundary of Ω, s is the arc-length along $\partial\Omega$, and $\hat{n}$ is the outside unit normal to $\partial\Omega$.

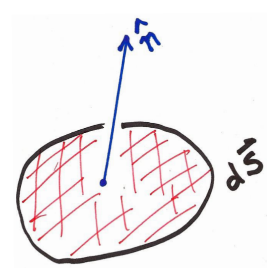
There are **two implicit assumptions** used above

1. The flux of conserved stuff is local: stuff does not vanish somewhere and re-appears elsewhere (this would not violate conservation). For most types of physical stuff this is reasonable. But one can think of situations where this is not true — e.g.: when you wire money, it disappears from your local bank account, and reappears elsewhere (with some loses due to fees, which go to other accounts).
2. The flux is given by a vector. But: $\Rightarrow$ *Why should this be so?* $\Leftarrow$ (8.2)
   The objective of this problem is to answer this question.

Given item **1**, **the flux can be characterized/defined as follows** †

For any surface element $d\vec{S}$ (at a point $\vec{x}$, with unit normal $\hat{n}$) the flux indicates how much stuff, per unit time and unit area, crosses $d\vec{S}$ from one side to the other, in the direction of $\hat{n}$. (8.3)

† This is in 3D. For a 2D, change "surface element" to "line element".



It follows that the flux should be a scalar function of position, time, and direction. That is:

$$q = q(\vec{x},\, t,\, \hat{n}), \tag{8.4}$$

[4] N in the range $20 \le N \le 200$ should be more than enough to see what happens.

where q is the amount of stuff, per unit time and unit length, crossing a curve [5] with unit normal $\hat{n}$ from one side to the other (with direction [6] given by $\hat{n}$). Then (8.1) takes the form

$$\frac{d}{dt}\int_{\Omega}\rho\,\mathrm{d}x_1\mathrm{d}x_2 = -\int_{\partial\Omega} q(\vec{x},\,t,\,\hat{n})\,\mathrm{d}s. \tag{8.5}$$

However, in this form we cannot use Gauss' theorem to transform the integral on the right over $\partial\Omega$, into one over Ω. *This is a serious problem, for this is the crucial step in reducing* (8.1) *to a pde.*

Your task: *Show that,* provided that ρ and q are "nice enough" functions (e.g.: continuous partial derivatives), *equation* (8.5) *can be used to show that q has the form*

$$q = \hat{n}\cdot\vec{q}, \tag{8.6}$$

for some vector valued function $\vec{q}(\vec{x},\,t)$.

Hints.

A. *It should be obvious that the flux going across any curve (surface in 3D) from one side to the other should be equal and of opposite sign to the flux in the opposite direction. That is, q in (8.4) satisfies*

$$q(\vec{x},\,t,\,-\hat{n}) = -q(\vec{x},\,t,\,\hat{n}). \tag{8.7}$$

Violation of this would result in the conserved "stuff" accumulating (or being depleted) at a finite rate from a region with zero area (zero volume in 3D), which is not compatible with the assumption that ρ_t is continuous and equation (8.5). **Note:** there are situations where it is reasonable to make models where conserved "stuff" can have a finite density on curves or surfaces (e.g.: surfactants at the interface between two liquids, surface electric charge, etc.). Dealing with situations like this requires a slightly generalized version of the ideas behind (8.7).

B. *Given an arbitrary small curve segment* [7] *of length $h > 0$ and unit normal $\hat{n}$, realize it as the hypotenuse of a right triangle where the other sides are parallel to the coordinate axes. Then write (8.5) for the triangle, divide the result by h, and take the limit $h \downarrow 0$. Note that, if the segment of length h is parallel to one of the coordinate axis, then one of the sides of the triangle has zero length, and the triangle has zero area — but the argument still works, albeit trivially (it reduces to the argument in* **A***).*

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9 Traveling waves for KdV and the Airy paradox

Statement: Traveling waves for KdV and the Airy paradox

Consider the Korteweg de-Vries (KdV) equation

$$u_t + u_x + 6\,u\,u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \tag{9.1}$$

where $u = u(x,\,t)$. Before getting to the problem's questions, some needed background.

1. It seems reasonable to assume that the small amplitude solutions are well represented by the linearized equation. In particular, that we can write

$$u \approx \int_{-\infty}^{\infty} F(k)\,e^{i\,(k\,x-\omega\,t)}\,dk, \quad \text{where} \quad \omega = k - k^3, \tag{9.2}$$

[5] In 3D: "... and unit area, crossing a surface ..."
[6] A positive q means that the net flow is in the direction of $\hat{n}$.
[7] You can assume it is straight, since a limit $h \downarrow 0$ will occur.

and F is the Fourier Transform of the initial data $u_0(x) = u(x, 0)$.

Imagine now that u_0 is a localized bump [8], write F in polar form [9] $F = f\, e^{i\,\phi_0}$, and rewrite (9.2) as

$$u \approx \int_{-\infty}^{\infty} f(k)\, e^{i\,\phi + i\,k\,x}\, dk, \quad \text{where} \quad \phi = \phi_0 - \omega\, t. \tag{9.3}$$

In particular

$$u_0 = \int_{-\infty}^{\infty} f(k)\, e^{i\,\phi_0 + i\,k\,x}\, dk. \tag{9.4}$$

The way a Fourier Transform can represent a localized function is through destructive interference. This requires a careful balance between the amplitude f and the angle ϕ_0. However

$$\frac{d^2\phi_0}{dk^2} - \frac{d^2\phi}{dk^2} = \frac{d^2\omega}{dk^2}\, t, \quad \text{where} \quad \frac{d^2\omega}{dk^2} \neq 0. \tag{9.5}$$

Thus the required balance is destroyed by the time evolution (see note below). It follows that

> *The solution in (9.3), ceases to be localized in space as time increases. Oscillations appear, which spread over an ever larger region (growing linearly with time). The bump "disperses".* (9.6)

Note. The reason that we need to look at the second derivative in (9.5) is that linear changes in the angle merely cause a shift in space of the transform. A wave system for which $\frac{d^2\omega}{dk^2} \neq 0$ is called *dispersive*, for the reasons explained above.

2. Yet equation (9.1) has steady traveling wave solutions (with constant speed s)

$$u = U(x - s\,t), \quad \text{where } u \text{ vanishes exponentially as } |x - s\,t| \to \infty. \tag{9.7}$$

Furthermore, these solutions satisfy $0 < U \leq a$, where the parameter $a = \max(U)$ can be any positive number — in particular, as small as you care it to be.

This is in clear contradiction with (9.6)! $\longleftarrow$ *The Airy "paradox".*

It is hard to argue with (9.7), which means that something is wrong with the argument leading to (9.6) — not too surprising since this argument is not "rigorous". Yet, there is no question that

$$\int_{-\infty}^{\infty} F(k)\, e^{i\,(k\,x - \omega\,t)}\, dk \tag{9.8}$$

exhibits the behavior in (9.6) — this is rigorous. Thus the problem must be with $\approx$ in (9.2). *The question is what, and is it something with a clean intuitive interpretation,* or is this a problem with a subtle and very technical, but not intuitive, mathematical reason behind it?

PROBLEM TASKS:

A. Write the solutions in (9.7), parameterized by a. This involves elementary functions only.

B. Study the way U scales with a, and provide an intuitive explanation for the Airy paradox.
Hint. What happens with the dispersive effects as a vanishes?

THE END

[8] In particular, u_0 decays very fast as $|x| \to \infty$. Say, exponentially.

[9] Here both f and ϕ_0 are real valued.