Problem Set Number 01, 18.306
MIT (Winter-Spring 2021)

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Due Fri. March 12, Spring 2021.
Turn it in via the canvas site website.

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1 Fundamental Diagram of Traffic Flow #03

Statement: Fundamental Diagram of Traffic Flow #03

Many state laws state that: for each 10 mph (16 kph) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 ft (5 m)). There is another law that gives a maximum speed limit (assume that this is 50 mph (80 kph)). **Find the flow of cars as a function of density, $q = q(\rho)$**, that results from these two laws.

The state laws on following distances stated in the prior paragraph were developed in order to prescribe a spacing between cars such that rear-end collisions could be avoided, as follows:

- **a.** Assume that a car stops instantaneously. *How far would the car following it travel* if moving at u mph and
  - **a1.** The driver's reaction time is τ, and
  - **a2.** After a delay τ, the car slows down at a constant maximum deceleration α.
- **b.** The calculation in part **a** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate α, but the driver in the following car still takes a time τ to react. *How far back does a car have to be, traveling at u mph, in order to prevent a rear-end collision?*
- **c.** Show that the law described in the first paragraph of this problem corresponds to part **b**, if the human reaction time is about 1 sec. and the length of a car is about 16 ft (5 m).

Note: *What part **c** is asking you to do is to justify/derive the state law prescription, using the calculations in part **b** to arrive at the minimum car-to-car separation needed to avoid a collision when the cars are forced to brake.*

2 Ill posed Laplace equation problem #01

Statement: Ill posed Laplace equation problem #01

Consider the following problem involving Laplace's equation

$$u_{xx} + u_{yy} = 0, \tag{2.1}$$

on the strip $0 < x < 2\pi$ and $-\infty < y < \infty$:

Given $u(0,\, y) = f(y)$ and $u_x(0,\, y) = g(y)$, determine $u(2\pi,\, y) = h(y)$ — where f and g are smooth periodic functions.

Show that this is an ill-posed problem.

Hint: Consider what happens with high frequency perturbations.

3 Laplace equation problem #01

Statement: Laplace equation problem #01

Consider a thin, homogeneous, heat conducting sheet, insulated on the top and the bottom. Then, if $T = T(x,\, y,\, t)$ is the temperature in the sheet, the conservation of heat (and Fick's law) leads to the heat equation — which in non-dimensional units has the form

$$T_t = \Delta\, T = T_{xx} + T_{yy}. \tag{3.1}$$

Let Ω be the region of space occupied by the sheet, and assume that along the boundary $\partial\Omega$ of this region the heat flux is known and given by some function, say: $F = F(s)$ per unit length (where s is the arc-length along $\partial\Omega$).

The problem to be solved is then (3.1) inside Ω, with the boundary conditions on $\partial\Omega$

$$\partial_n T = \hat{n} \cdot \nabla T = F(s), \quad \text{where } \hat{n} = \text{unit outside normal to } \partial\Omega. \tag{3.2}$$

In particular, for *steady state*, we have Laplace's equation in Ω: $$0 = \Delta\, T = T_{xx} + T_{yy}, \tag{3.3}$$
with the *Neumann* boundary condition in (3.2).

1. Show that there is an integral condition that F must satisfy if the problem (3.2–3.3) has a solution. **Hint:** *Gauss theorem.*
2. Give a physical interpretation to the condition in **1.** Why do you need it, and what happens when it is not satisfied?
3. The solution to (3.2–3.3), if there is one, is determined only up to an arbitrary additive constant. How would you determine this constant, and what is it related to — i.e.: knowledge of what physical quantity gives it to you?

4 Nonlinear solvable ODEs

4.1 Statement: A nonlinear solvable ode #01

Generally nonlinear ode, even simple scalar and first order ones, do not admit explicit exact solutions. There are, however, some exceptions. For example, consider the ode

$$0 = 1 + \frac{1}{y'} + \left(x - \frac{y}{y'}\right)^2, \tag{4.1}$$

where $y' = \dfrac{dy}{dx}$. This ode is nonlinear with variable coefficients, and not separable. Yet it can be solved exactly, even though the standard methods do not apply. Your **task** here is to do this.

Hint. Introduce the variables,

$$u = \frac{1}{y'} \quad \text{and} \quad v = x - \frac{y}{y'} = x - y\,u, \tag{4.2}$$

and investigate what equation (4.1) implies for them.

5 Small vibrations of a 2D string under tension (sv2ds)

5.1 Introduction: Small vibrations of a 2D string under tension

Here you are asked to derive *equations for the transversal vibrations of a thin elastic string under tension*, under several scenarios. The following hypotheses are common to all of them (5.1)

1. The string is homogeneous, with *mass density* (mass per unit length) $\rho =$ constant.
2. The motion is restricted to the x-y plane. At equilibrium the string is described by $y = 0$ and $0 \leq x \leq L =$ string length. The *tension* $T > 0$ is then *constant*. See remark **5.1**, #1–3.
3. The string has no bending strength.
4. The amplitude of the vibrations is very small compared with their wavelength, and any longitudinal motion can be neglected. Thus:
   — The string can be described in terms of a *deformation function* $u = u(x,\, t)$,
   such that the equation for the *curve describing the string* is[1] $y = u(x,\, t)$.
   — The tension remains constant throughout — see remark **5.1**, #4.

[1] In this approximation the x-coordinate of any mass point on the string does not change in time.

Remark 5.1 *Some details:*

#1 *We idealize the string as a curve. This is justified as long as $\lambda \gg d$, where λ = scale over which motion occurs, and d = string diameter. This condition justifies item* **3** *as well.*

#2 *The tension is generated by the elastic forces (assume that the string is stretched). At a point along the string, the tension is the force with which each side (to the right or left of the point) pulls on the other side,*[2] *and it is directed along the direction tangent to the string (there are no normal forces, see item* **3***).*

#3 *At equilibrium the tension must be constant. For imagine that the tension is different at two points $a < b$ along the string. Then the segment $a \leq x \leq b$ would receive a net horizontal force (the difference in the tension values), and thus could not be at equilibrium.*

#4 *The hypotheses imply that string length changes can be neglected. Thus the stretching generating the tension does not change significantly, and the tension remains equal to the equilibrium tension T.*

5.2 Statement: sv2ds04 (string on elastic bed)

Here we consider the simple situation, where (on the string) there is no other force beside the tension T, nor any dissipation. However, the string is attached to an elastic bed, which produces an elastic restoring force (per unit length) pulling the string towards its equilibrium position. Thus this force per unit length has the simple form

$$-\boldsymbol{k_b}\,\boldsymbol{u}, \tag{5.2}$$

where $\boldsymbol{k_b} > \boldsymbol{0}$ is the spring bed constant. Under these conditions, plus those in § 5.1:

Task #1. Use the conservation of the transversal momentum to derive an equation for $\boldsymbol{u}$. Thus:

i) Obtain a formula for the transversal momentum density along the string (momentum per unit length).

ii) Obtain a formula for the transversal momentum flux along the string (momentum per unit time).

iii) Use the differential form of the conservation of transversal momentum to write a pde for u.
Note that there is a momentum source as well.

Hint. Remember that internal string forces[3] *are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces.* Be careful with the sign here! *Remember also that u_x is very small.*

Task #2. Use the conservation of energy to derive another equation for $\boldsymbol{u}$. Furthermore, show that the solutions to the equation derived in task #1 satisfy the conservation of energy equation.

Hint. The energy density is the sum of the kinetic energy per unit length (e_k), the elastic energy per unit length stored in the bed (e_b), and the elastic energy per unit length stored in the string deformation (e_d). Expressions for the first two are fairly easy to obtain. Let us now consider how to get an expression for e_d. In fact, you only need to compute the difference between the elastic energy, and some constant energy — specifically: the elastic energy of the string at equilibrium. But the tension is constant (see § 5.1) over the whole stretching process taking the string from equilibrium $y = 0$ to $y = u(x, t)$. Thus $e_d\,dx$ is the product of T times the change in length of the segment dx. All you need now is the formula for the change in length of any interval dx (actually, just the leading order approximation to this, in the u_x small limit).

This yields $\boldsymbol{e_d = T\,\left(\sqrt{1+u_x^2}-1\right) = \frac{1}{2}\,T\,u_x^2}$, since the arc-length is $ds = \sqrt{1+u_x^2}\,dx$, and u_x is small.

Task #3. Assuming that the string is tied at both ends $\boldsymbol{u(0,\,t) = u(L,\,t) = 0}$, write an equation for the evolution of the total energy. That is: $\boldsymbol{\frac{d\mathcal{E}}{dt}}$ =??

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[2] If you were to cut the string, this would be the force needed to keep the lips of the cut from separating.

[3] Internal forces = forces by one part of the string on another.

6 Well and/or ill posed ODE #01

Statement: Well and/or ill posed ODE #01

Consider the two ode problems [4]

$$\frac{dx}{dt} = \sigma \,\mathrm{sign}(x)\,\sqrt{|x|} \quad \text{and} \quad x(0) = 0, \tag{6.1}$$

where either $\sigma = 1$, or $\sigma = -1$. One of these problems is well posed, and the other is ill posed. **Show this.**

Hint #1. To get some intuition, draw the the slope field in the x versus t plane.

Hint #2. Consider the solutions obtained by separating variables. Are there any other solutions?

Hint #3. Note that, if:

(a) $x = x_1(t)$ solves the ode for $0 \leq t \leq t_0$,

(b) $x = x_2(t)$ solves the ode for $t_0 \leq t$, and

(c) $x_1(t_0) = x_2(t_0)$,

Then $x(t) = x_1(t)$ for $0 \leq t \leq t_0$, and $x(t) = x_2(t)$ for $t_0 \leq t$, solves the ode for all $t \geq 0$.

7 Well and/or ill posed ODE #02

Statement: Well and/or ill posed ODE #02

Consider a **cylindrical water bucket,** loosing its contents through a small, but not too small,[5] hole in the bottom. Assume also that no water is being added at the top.

In this problem you are asked to:

1. **Derive an equation** for the height of the water level in the bucket.
2. **Give a physical interpretation** to the results from the exercise in §6. In particular, show that the lack of uniqueness is natural and obvious, not pathological.

Let $h = h(t) =$ water height in the bucket, $a =$ small hole cross-sectional area, $A =$ bucket cross-sectional area (constant), $v = v(t) =$ water velocity through the hole, from inside to outside $[v \geq 0]$, $\rho =$ water density, and $g =$ acceleration of gravity. Now **derive an ode for h**, as follows:

a. Use the conservation of the water volume to write an equation that relates $\dot{h}$ to v.

b. Use the conservation of energy to write an equation that relates v to h. Proceed as follows:

1. Write a formula for the gravitational potential energy, V, of the water in the bucket.
2. Write a formula for the kinetic energy flux, K_f, produced by the stream of water through the hole at the bottom.
3. Neglect dissipation, so that all the potential energy loss (as the water level falls) is converted into kinetic energy.

This will yield the desired equation.

c. Alternatively, obtain an equation relating v to h as follows **(do BOTH b and c)**

1. Assume that the pressure in the bucket is hydrostatic.
2. Assume Bernoulli's law, so that $p = p_a + \frac{1}{2}\rho v^2$, where p is the pressure at the bottom of the bucket and p_a is the pressure outside the bucket. [6]

These assumptions, as well as those in **b**, are crucially dependent on the hole being small.

[4] Note that the ode right hand side is a continuous function of x, in spite of the sign function present there.
[5] If the hole is too small, viscosity and surface tension become important. Here we neglect these, as well as evaporation.
[6] Assume that p_a is constant.

d. Combine **a** and **b** to get the desired ode.

Compare the ode you just obtained with (6.1), and **give a physical interpretation to the two cases considered in** § 6 — i.e. $\sigma = \pm 1$.

8 Well and/or ill posed PDE #02

Statement: Well and/or ill posed PDE #02

Let $\rho = \rho(x,\, t)$ and $q = q(x,\, t)$ be the density (and corresponding flux) for some conserved quantity. Then, in the absence of sources:

$$\rho_t + q_x = 0, \quad -\infty < x < \infty. \tag{8.1}$$

Assume that, while examining the physical problem leading to this equation, you conclude that a "good approximation" for the flux is

$$q = \rho + c\,\rho_{xxx}, \tag{8.2}$$

where c is some constant. Substituting (8.2) into (8.1) yields a pde for ρ. *What restriction should you impose on the constant c so that this pde is not ill-posed?* Specifically: what restriction on c guarantees that the equation does not exhibit arbitrarily large growth factors at high frequencies?
Hint. *Examine the behavior of sinusoidal in space solutions: $\rho \propto e^{i\,k\,x}$ with k a real constant.*

9 Well and/or ill posed PDE #05

Statement: Well and/or ill posed PDE #05

Let $\rho = \rho(x,\, t)$ and $q = q(x,\, t)$ be the density (and corresponding flux) for some conserved quantity. Then, in the absence of sources:

$$\rho_t + q_x = 0, \quad -\infty < x < \infty. \tag{9.1}$$

While examining the physical problem leading to this equation, you conclude that "good approximations" for the density and flux are

$$\rho = u_t \quad \text{and} \quad q = u_x - c\,u_{x\,x\,x} - 2\,u_{xt}, \tag{9.2}$$

where c is a constant, and $u = u(x,\, t)$. Substituting (9.2) into (9.1) yields a pde for u. *What restriction should you impose on c so that this pde is not ill-posed?* Specifically: what restriction guarantees that the equation does not exhibit arbitrarily large growth factors at high frequencies?
Hint. *Examine the behavior of sinusoidal in space solutions: $u \propto e^{i\,k\,x}$ with k a real constant.*
Warning: *To have an ill-posed situation, arbitrarily large growth factors are needed. Growth alone, if bounded, is not enough — this is a sign of some instability, not the same as being ill-posed.*

THE END