Problem Set Number 02, 18.306
MIT (Winter-Spring 2018)
Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)
March 20, 2018

Tue April 3, 2018.
Turn it in (by 3PM) at the Math. Problem Set Boxes, right outside ................. room 4-174. There is a box/slot there for 306. Be careful to use the right box (there are many slots).

Contents

1  Impedance matching (extra task) ........................................... 1

2  Traveling wave solutions and shocks (BuHe01) ............... 1

   Check if an hypothetical model of traffic flow makes sense ......................... 1

3  Linear 1st order PDE (problem 09) ................................. 2

   Surface evolution. Solve \( h_t = (A/r) h_r \) ............................. 2

4  Linear 1st order PDE (problem 12) ............................... 3

   Investigate IBVP along curve where solution is not smooth: \( u_t + \frac{1+x}{1+t} u_x = 0 \) on \( x, t > 0 \) ........... 3

5  IVP for a kinematic wave equation #03 .......................... 4

   IVP problem for \( u_t + (0.5 u^2)_x = 0 \) ............................. 4

6  Discontinuous Coefficients in Linear 1st order pde #02 ........... 4

1 Impedance matching (extra task)

Do the extra task in § 8.3.2 of the answer to problem set #01

2 Traveling wave solutions and shocks (BuHe01)

Statement: Traveling wave solutions and shocks (BuHe01)

Imagine that someone tells you that the following equation is a model for traffic flow:

\[ c_t + c c_x = \nu c_{xt}, \]

(2.1)

where \( \nu > 0 \) is “small”\(^1\) and \( c \) is the the wave velocity — related to the car density via \( c = \frac{dQ}{d\rho} \). The objective of this problem is to ascertain if (2.1) makes sense as model for Traffic Flow. To this end, answer the two questions below.

Question #1. Does the model have acceptable traveling wave “shock” solutions ...................... \( c = F(z) \),

where ................................................................................................................. \( z = \frac{x-U t}{\nu} \),

and \( U \) is a constant?

\(^1\) Note that \( \nu \) has dimensions of inverse length, so small means compared with some appropriate length scale.
Here “acceptable” means the following

1a. The function $F$ has finite limits as $z \to \pm \infty$, i.e.: $c_L = \lim_{z \to -\infty} F(z)$ and $c_R = \lim_{z \to +\infty} F(z)$.

Further: the derivatives of $F$ vanish as $z \to \pm \infty$, and $c_L \neq c_R$.

This means that, as $\nu \to 0$, the solution $c$ becomes a discontinuity traveling at speed $U$, with $c = c_L$ for $x < Ut$ and $c = c_R$ for $x > Ut$. That is, a shock wave.

1b. The solution satisfies the Rankine-Hugoniot jump conditions

$$U = \frac{[Q]}{[\rho]}$$

where $\rho_L$ and $\rho_R$ are related to $c_L$ and $c_R$ via $c_L = \frac{dQ}{d\rho}(\rho_L)$ and $c_R = \frac{dQ}{d\rho}(\rho_R)$.

Assume that $Q = Q(\rho)$ is a quadratic traffic flow function — see remark 2.1.

1c. The solution satisfies the Entropy condition

$$c_L > U > c_R.$$

To answer this question:

A. Find all the solutions satisfying 1a. Get explicit formulas for $F$ and $U$ in terms of $c_L$, $c_R$, and $z$.
B. Check if the solutions that you found satisfy 1b.
C. Check if the solutions that you found satisfy 1c.
D. Finally, given A-C: Does. so far, the equation make sense as a model for traffic flow?

Hints.

- Find the ode $F$ satisfies. Show it can be reduced to the form $F' = P(F)$, where $P$ = second order polynomial.
- Write $P$ in terms of its two zeroes, $c_1$ and $c_2$, and express all the constants (e.g.: $U$) in terms of $c_1$ and $c_2$.
- Solve now the equation, and relate $c_1$ and $c_2$ to $c_L$ and $c_R$. You are now ready to proceed with A-D.
- Remember that, while the density $\rho$ has to be non-negative, wave speeds can have any sign.

Question #2. As a second model check, study the small perturbations from a constant state $c = c_0$. Let $c = c_0 + u$, where $u$ is “infinitesimal”. Write the equation for $u$ and look for solutions of the form

$$u = e^{ikx + \lambda t},$$

where $-\infty < k < \infty$, and $\lambda$ is some function of $k$.

How do these solutions behave? Is this reasonable for a traffic flow model?

Remark 2.1 An attempt at “justifying” (2.1) goes as follows:

It is not unreasonable to assume that the drivers not only respond to the local traffic density, but its rate of change as well. A simple way to model this is to write

$$q = Q(\rho) - \nu \rho_t,$$

for the flow rate $q = q(x, t)$, where $\nu > 0$ is a constant characterizing the drivers response to the local rate of change in the density (the reason $\nu$ should be positive, is that the normal driver’s reaction to a density increase should be to slow down).

Substituting (2.2) into the equation for conservation of cars, yields:

$$\rho_t + c(\rho) \rho_x = \nu \rho_{tx},$$

where $c = \frac{dQ}{d\rho}$. When $Q$ is a quadratic function of $\rho$, $c$ is a linear function of $\rho$ and (2.3) is equivalent to (2.1).

3 Linear 1st order PDE (problem 09)

Statement: Linear 1st order PDE (problem 09)

Surface Evolution. The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of
its “height” $h = h(x, y, t)$ relative to the $(x, y)$-plane of reference. Under appropriate conditions, a rather complicated PDE can be written\(^2\) for $h$. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r} h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \quad (3.1)$$

Axial symmetry is assumed, so that $h = h(r, t)$. Obviously, $h$ should be an even function of $r$. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r, 0) = H(r^2), \quad (3.2)$$

where $H$ is a smooth function describing a localized bump. Specifically: \(i\) $H(0) > 0$, \(ii\) $H$ is monotone decreasing, \(iii\) $H \to 0$ as $r \to \infty$. Note that $h(r, 0)$ is an even function of $r$.

1. Using the theory of characteristics, write an explicit formula for the solution of (3.1 – 3.2).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2A$?
4. Show that the resulting solution is an even function of $r$ for all times.
5. Show that, as $t \to \infty$, the bump shrinks and vanishes. Hint. Pick some example function $H$ with the properties above, and plot the solution for various times. This will help you figure out why the bump shrinks and vanishes.

### 4 Linear 1st order PDE (problem 12)

**Statement:** Linear 1st order PDE (problem 12)

Consider the following problem, in the region $x, t > 0$,

$$u_t + \frac{1+x}{1+t} u_x = 0, \quad \text{with data} \quad u(x, 0) = \frac{1}{1+x} \quad \text{and} \quad u(0, t) = \frac{1}{1+t}. \quad (4.1)$$

1. Use the method of characteristics to solve the problem. Write explicit formula(s) for $u$ for all $x, t > 0$.
2. Use the result in 1 to show that the solution is smooth (infinite partial derivatives) in the region $x, t > 0$, except along a special curve $\Gamma$ (see item 3). Describe $\Gamma$.
3. Is the solution continuous for points on $\Gamma$? What happens with $u_t$ and $u_x$ for points on $\Gamma$?
4. For the solution $u = u(x, t)$ found in item 1, does the pde\(^3\) in (4.1) make sense for points on $\Gamma$? In other words, does $u$ satisfy the pde everywhere in $x, t > 0$?

---

\(^2\)From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.

\(^3\)That is: $u_t + \frac{1+x}{1+t} u_x = 0$. 
5 IVP for a kinematic wave equation #03

Statement: IVP for a kinematic wave equation #03

Let $u$ be the density of some conserved quantity, governed by the equation

$$u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad -\infty < x < \infty,$$

(5.1)

where shocks should be used to avoid multiple values. Consider now the following initial value $u(x, 0) = -\sin(x)$, and answer/perform the following questions/tasks:

1. **When and where do shocks form?** Find the places in space-time at which a shock has to be started to prevent the characteristics from crossing, and stop multiple values from arising.

2. **Where are the shocks located, for all times? Explicit formula required.** Find the shock path, for each shock identified in item 1. Hint: Do not attempt to explicitly compute $u$ on each side of the shock, to then solve the shock equations. Instead, show (e.g.: using the characteristic equations) that a certain symmetry applies to the solution. Then use it to get the shock position.

3. **What does the solution look like as $t \to \infty$?** Give a complete, and explicit, description — valid with errors much smaller than $1/t$. Hint: The characteristics end when they encounter a shock. Thus, argue that: as $t \to \infty$, only the characteristics starting near certain points remain in the solution. Use this to get an approximate description of the solution in this limit.

4. **Extra credit:** Calculate the first correction to the approximation in item 1.

5. **Find (a parametric description of) the envelope of the characteristics for $t > 0$, and plot it.** Furthermore: (5.1) Note that the envelope has many branches, describe them all. (5.2) Describe the asymptotic behavior of each branch for $t \to \infty$.

Hint. The initial value is periodic of period $2\pi$. Argue that this period should carry over to the characteristics — hence to the solution, shocks, envelope of the characteristics, etc. Thus you only have to describe one copy only of each of the objects that you are asked to analyze in this problem. Important: When I say “argue” above, I mean it. Using this just because I said so here is not acceptable.

6 Discontinuous Coefficients in Linear 1st order pde #02

Statement: Discontinuous Coefficients in Linear 1st order pde #02

Singularities (in particular, discontinuities) in the coefficients of a pde can create ambiguities in the meaning of the equation. Sometimes these ambiguities can be easily resolved, and other times they cannot. In all cases, however, it is advisable to go back to the physical system that the pde is supposed to model, and either (a) Check that the meaning given to the solutions across the singularities in the coefficients makes physical sense, or (b) Seek for the meaning, if not clear, there.

In this exercise we consider an example of the situation described in the prior paragraph. The task is to give a unique, unambiguous, meaning to the following initial value problem (I.V.P.)

$$u_t + \text{sign}(x) u_x = g(x) \quad \text{for } t > 0 \text{ and } -\infty < x < \infty, \quad \text{with } u(x, 0) = U(x),$$

(6.1)

---

4 When I say argue, I really mean, argue. Using this just because I said so is not acceptable.

5 What does it mean to be a solution?
where $g$ and $U$ are “arbitrary” smooth functions. An important (practical) consideration is that well posed questions do not have answers sensitive to small changes in problem formulation. Hence

The solution to the problem in (6.1) should not change very much \{for any finite time interval $0 < t < T$\} if either $g$, or the coefficient function $\text{sign}(x)$, are modified slightly.

We use this requirement to give a clear meaning to the problem in (6.1), as follows.

**STEP 1.** Replace (6.1) by the set of problems, parameterized by $\epsilon > 0$,

$$u_t + f_\epsilon(x)u_x = g(x) \text{ for } t > 0 \text{ and } -\infty < x < \infty, \text{ with } u(x, 0) = U(x),$$

(6.3)

where $f_\epsilon$ is a smooth, non-decreasing, function satisfying $f_\epsilon(x) = \text{sign}(x)$ for $|x| > \epsilon$. **Show that the limit $\epsilon \downarrow 0$ of the solutions to (6.3) exists, and it is independent of the choice of the functions $f_\epsilon$.**

**Hint:** write the problem (6.3) in characteristic form, and consider what happens with the characteristics as $\epsilon \downarrow 0$. Drawing them in the space time diagram should be helpful.

Thus we can use the $\epsilon \downarrow 0$ limit of (6.3) to give a clear meaning to (6.1).

**Hint/Warning:** be careful with the limit in the region $|x| \leq t$! The best formulation of the limit is as **two signaling/initial value problems:** one for $x, t > 0$ and another one for $x < 0 < t$ — each with appropriate boundary conditions on $x = 0, t > 0$.

**STEP 2.** Show that the solution obtained in step 1 is not sensitive to small changes in the functions $g = g(x)$ or $U = U(x)$.

THE END