Answers to Problem Set # 04, 18.306
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1 Fundamental solution for the Laplace operator

1.1 Statement: Fundamental solution for the Laplace operator

Let Δ be the Laplace operator in $\mathcal{R}^d$

$$\Delta = \sum_{n=1}^{d} \partial_{x_n}^2, \tag{1.1}$$

where $d \geq 3$. Further, let G be defined by

$$G = \frac{1}{(d-2)\,\alpha_d\, r^{d-2}}, \quad \text{where} \quad r = \sqrt{\sum_{n=1}^{d} x_n^2} \tag{1.2}$$

and $\boldsymbol{\alpha_d}$ is **the "area" of the unit sphere $\boldsymbol{S^d(1)}$ in $\boldsymbol{d}$-dimensions. Show that**

$$\boldsymbol{\Delta G = -\delta(\vec{x})}, \tag{1.3}$$

where δ is Dirac's delta in $\mathcal{R}^d$. That is: show that *for any test function*[1] ψ *in* $\mathcal{R}^d$

$$\psi(0) + \int_{\mathcal{R}^d} G\,\Delta\psi\,dV = 0, \tag{1.4}$$

where $dV = dx_1 \ldots dx_d$ is the "volume" element in $\mathcal{R}^d$. Proceed as follows

1. For any function f in R^d, define the radial function $\overline{f} = \overline{f}(r)$ by averaging over the angular variables:

$$\overline{f} = \frac{1}{\alpha_d} \int_{S^d(1)} f(r, \vec{\sigma})\,dS = \frac{1}{\alpha_d\, r^{d-1}} \int_{S^d(r)} f\,d\tilde{S}, \tag{1.5}$$

where: (i) $\vec{\sigma}$ are the angular variables in R^d; (ii) dS is the surface element in $S^d(1)$ — dS involves only the angular variables; and (iii) $d\tilde{S} = r^{d-1}\,dS$ is the area element in $S^d(r)$ (sphere of radius r).

2. Use Gauss theorem to express $\overline{\psi}_r$ in terms of the integral of $\Delta\psi$ over a ball of radius r. That is, in terms of: $\displaystyle\int_{B^d(r)} \Delta\psi\,dV$, where $B^d(r) = \{\vec{x} \in R^d \mid \|\vec{x}\| \leq r\}$.

3. Use **2** to show that

$$\Delta\overline{\psi} = \overline{\Delta\psi}. \tag{1.6}$$

Note: $\Delta g = r^{1-d}\,(r^{d-1}\,g_r)_r$ for radial functions.

4. Use the fact that $dV = r^{d-1}\,dr\,dS$ to re-write the integral in (1.4). You are now a small step away from being able to show (1.4) applies.

Important: You **do not need** explicit expressions for the angular variables.
This problem involves very few grungy calculation.

1.2 Answer: Fundamental solution for the Laplace operator

Since $\psi_r = \hat{n}\,\mathrm{grad}(\psi)$, where $\hat{n}$ is the radial unit vector, we have

$$\overline{\psi}_r = \frac{1}{\alpha_d} \int_{S^d(1)} \psi_r(r, \vec{\sigma})\,dS = \frac{1}{\alpha_d\, r^{d-1}} \int_{S^d(r)} \hat{n}\,\mathrm{grad}(\psi)\,d\tilde{S} = \frac{1}{\alpha_d\, r^{d-1}} \int_{B^d(r)} \Delta\psi\,dV \tag{1.7}$$

where $S^d(r)$ is the sphere of radius r, $d\tilde{S} = r^{d-1}\,dS$ is the area element in $S^d(r)$, and $B^d(r)$ is the ball of radius r. We now use (1.7) to write

$$\begin{aligned} \Delta\overline{\psi} &= \frac{1}{r^{d-1}} \left(r^{d-1}\,\overline{\psi}_r\right)_r = \frac{1}{\alpha_d\, r^{d-1}} \left(\int_{B^d(r)} \Delta\psi\,dV\right)_r \\ &= \frac{1}{\alpha_d\, r^{d-1}} \left(\int_0^r \rho^{d-1}\,d\rho \int_{S^d(1)} \Delta\,\psi(\rho, \vec{\sigma})\,dS\right)_r = \overline{\Delta\psi}. \end{aligned} \tag{1.8}$$

Finally, we can write

$$\begin{aligned} \int_{\mathcal{R}^d} G\,\Delta\psi\,dV &= \frac{1}{(d-2)\,\alpha_d} \int_0^\infty r\,dr \int_{S^d(1)} \Delta\,\psi(r, \vec{\sigma})\,dS = \frac{1}{d-2} \int_0^\infty r\,dr\,\overline{\Delta\psi} \\ &= \frac{1}{d-2} \int_0^\infty r^{2-d}\,\left(r^{d-1}\,\overline{\psi}_r\right)_r\,dr = \int_0^\infty \overline{\psi}_r\,dr = -\overline{\psi}(0) = -\psi(0). \end{aligned} \tag{1.9}$$

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[1] A smooth function decaying as fast as you need as $r \to \infty$.

2 Green's functions #11

2.1 Statement: Green's functions #11

The Green's function for the heat equation half line Dirichlet [2] signaling problem is defined by

$$G_t = G_{xx}, \quad \text{for} \quad -\infty < t < \infty \quad \text{and} \quad x > 0, \tag{2.1}$$

with the boundary condition

$$G(0,\, t) = \delta(t), \tag{2.2}$$

where $\delta(\cdot)$ denotes Dirac's delta function. In addition: G is bounded away from the origin $(0,\, 0)$ and satisfies **causality** $\boldsymbol{G = 0}$ **for** $\boldsymbol{t < 0}$.

Thus **we only need to find** $\boldsymbol{G}$ **for** $\boldsymbol{t \geq 0}$ **only.**

For any constant $\nu \neq 0$, $\boldsymbol{\nu^2\, \delta(\nu^2\, t) = \delta(t)}$. Thus: If G is a solution, $\nu^2\, G(\nu\, x,\, \nu^2\, t)$ is a solution.

Hence, from uniqueness, for any $\nu \neq 0$,

$$G(x,\, t) = \nu^2\, G(\nu\, x,\, \nu^2\, t). \tag{2.3}$$

Set $\nu = x/t$ to get, for some function $g = g(\xi)$,

$$G(x,\, t) = \frac{1}{t}\, g\left(\frac{x^2}{t}\right). \tag{2.4}$$

Notice that

A. $\boldsymbol{g(0) = 0}$. This follows because, for any $t > 0$ fixed, G must vanish as $x \downarrow 0$.

B. $\displaystyle \boldsymbol{\int_0^\infty g(\xi)\, \frac{d\xi}{\xi} = 1.}$ We have $\displaystyle \int_{-\infty}^\infty G(x,\, t)\, dt = \int_0^\infty g(\xi)\, \frac{d\xi}{\xi} =$ constant, for any $x > 0$.

By taking $x \downarrow 0$, and using (2.2), the result follows.

– **Task 1:** Substitute (2.4) into (2.1), and get an o.d.e. for g.

– **Task 2:** Solve the o.d.e. for g, and thus find the Green's function.

Hints: (i) *The o.d.e. for g is second order. It can be integrated once, to yield a first order equation.* **(ii)** *The general solution to the o.d.e. in* **1** *has two constants. These follow from* **A – B** *above.*

Remark 2.1 *For any fixed $x > 0$, the formula in (2.4) should, as $t \downarrow 0$, smoothly match with the solution $G \equiv 0$ for $t < 0$. This will be guaranteed by the fact that $g(\xi)$ vanishes exponentially as $\xi \to \infty$. Hence, G as given by (2.4), as well as all its derivatives, vanish as $t \downarrow 0$ for any $x > 0$.* ♣

2.2 Answer: Green's functions #11

Substituting (2.4) into (2.1) yields

$$4\,\xi\, \frac{d^2 g}{d\xi^2} + (2 + \xi)\, \frac{dg}{d\xi} + g = 0. \tag{2.5}$$

Hence, for some constant a,

$$4\,\xi\, \frac{dg}{d\xi} + (\xi - 2)\, g = 4\, a. \tag{2.6}$$

Thus, for some constant b,

$$g = a\left(\sqrt{\xi}\, \int_0^\xi \frac{e^{s/4} - 1}{s^{3/2}}\, ds - 2\right) e^{-\xi/4} + b\, \sqrt{\xi}\, e^{-\xi/4}. \tag{2.7}$$

Since $g(0) = 0$, $a = 0$. So:

$$g = b\, \sqrt{\xi}\, \exp\left(-\frac{\xi}{4}\right) \tag{2.8}$$

Then **B** yields

$$1 = b\, \int_0^\infty e^{-\xi/4}\, \frac{d\xi}{\sqrt{\xi}} = 2\, b\, \sqrt{\pi}, \tag{2.9}$$

which leads to the **final answer**

$$\boldsymbol{G = \frac{x}{2\, \sqrt{\pi}\, t^{3/2}}\, \exp\left(-\frac{x^2}{4\, t}\right).} \tag{2.10}$$

[2] Temperature prescribed on the boundary.

To visualize how this approaches a Delta function as $x \downarrow 0$, write G in terms of $\boldsymbol{\tau = \frac{4\,t}{x^2}}$. Then

$$\boldsymbol{G = \frac{4}{\sqrt{\pi}\,x^2}\,\frac{1}{\tau^{3/2}}\,\exp\left(-\frac{1}{\tau}\right)}. \tag{2.11}$$

As a function of τ increasing, G starts at 0 — with all its derivatives vanishing. It then raises rapidly to reach its maximum at $\tau = 2/3$, and then decays back to zero as $\tau \to \infty$. This function shape is then amplified and compressed by x^2 — via the pre-factor and the dependence of τ on x^2.

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3 Laplace equation in a circle #01

3.1 Statement: Laplace equation in a circle #01

Green's function: Laplace equation in a circle, with Dirichlet BC. Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) are insulated. This problem can be written — using polar coordinates and non-dimensional variables (selected so that the plate has radius 1) — in the form

$$\frac{1}{r^2}\,(r\,(r\,T_r)_r + T_{\theta\theta}) = \Delta\,T = 0, \quad \text{for } 0 \le \theta \le 2\pi \quad \text{and } 0 \le r \le 1, \tag{3.1}$$

with the boundary condition † $\qquad T(1,\,\theta) = h(\theta),$ for some given function h. (3.2)

† T and h are $2\,\pi$-periodic in θ.

The solution to this problem, as follows from separation of variables, is [3]

$$T = \sum_{n=-\infty}^{\infty} h_n\,r^{|n|}\,e^{i\,n\,\theta}, \quad \text{where} \quad h_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} h(\theta)\,e^{-i\,n\,\theta}d\theta. \tag{3.3}$$

For $r < 1$ this formula provides a solution to the equation even if h is not very smooth (or in fact, not even a function) and thus has a very poorly convergent Fourier series $\sum h_n\,e^{i\,n\,\theta}$. For example, assume that the sequence $\{h_n\}$ is bounded. Then, in any disk of radius $R < 1$: (i) the series for T in (3.3) converges absolutely; and (ii) the series obtained by term by term differentiation, for any derivative of T, also converges absolutely. It follows that T is smooth for $r < 1$. In a similar fashion, one can justify exchanging the order of integration and summation — as you are asked to do below.

1. Assume that $h = \delta(\theta)$, and **write an explicit formula for the solution T.**
2. In (3.3), substitute the formula for h_n into the series for T, exchange the summation and integration order, and

   derive an equation of the form $$T = \int_0^{2\,\pi} G(r,\,\theta-\phi)\,h(\phi)\,d\phi. \tag{3.4}$$
3. **Show that:**

   (a) $G > 0$ for $r < 1$. (b) $G(r,\,\theta) \to 0$ as $r \to 1$, if $\theta \ne 0$. (c) $\int_0^{2\,\pi} G(r,\,\theta)\,d\theta = 1$ for $r < 1$.

Hint. Add the positive and negative indexes n separately. Then use that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$.

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[3] Check, by direct substitution, that this is indeed the solution. That (3.2) is satisfied should be "obvious".

3.2 Answer: Laplace equation in a circle #01

Let us first **assume that $h = \delta(\theta)$.** Then $h_n = \dfrac{1}{2\pi}$ for all n, so that

$$2\pi T = \sum_{n=0}^{\infty} r^n e^{i n \theta} + \sum_{n=0}^{\infty} r^n e^{-i n \theta} - 1 = \frac{1}{1 - r e^{i\theta}} + \frac{1}{1 - r e^{-i\theta}} - 1.$$

Thus

$$T = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = G(r,\, \theta), \tag{3.5}$$

where G is defined by the equation.

More generally, $h_n = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} h(\phi)\, e^{-i n \phi}\, d\phi$ leads to

$$2\pi T = \int_0^{2\pi} \left(\sum_{n=0}^{\infty} r^n e^{i n (\theta - \phi)} + \sum_{n=0}^{\infty} r^n e^{-i n (\theta-\phi)} - 1 \right) h(\phi)\, d\phi.$$

Thus the general formula

$$T = \int_0^{2\pi} G(r,\, \theta - \phi)\, h(\phi)\, d\phi, \tag{3.6}$$

where G is as in (3.5).

Finally

(a) $1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1 - \cos\theta) > (1-r)^2$. Thus $G > 0$ for $r < 1$.

(b) $1 - 2r\cos\theta + r^2 = (r - \cos\theta)^2 + 1 - \cos^2\theta > 1 - \cos^2\theta$. Thus $\lim_{r\to 1} G(r,\, \theta) = 0$ if $\theta \neq 0$.

(c) For $r < 1$, $G = \dfrac{1}{2\pi}\left(\displaystyle\sum_{n=0}^{\infty} r^n e^{i n \theta} + \sum_{n=0}^{\infty} r^n e^{-i n \theta} - 1 \right)$, which converges absolutely, and can be integrated term by term. Thus $\displaystyle\int_0^{2\pi} G(r,\, \theta)\, d\theta = 1$ for $r < 1$.

In fact, the same argument in item c shows that, for $r < 1$, $\displaystyle\int_0^{2\pi} G(r,\, \theta)\, h(\theta)\, d\theta = \sum_{-\infty}^{\infty} r^{|n|}\, h_n.$

If h is smooth enough $h(0) = \displaystyle\sum_{-\infty}^{\infty} h_n$, where convergence is absolute. Thus **$G \to \delta(\theta)$ as $r \to 1$.**

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4 Laplace equation in a circle #02

4.1 Statement: Laplace equation in a circle #02

Green's function: Laplace equation in a circle, with Neumann BC. Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature flux is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) are insulated. This problem can be written — using polar coordinates and non-dimensional variables (selected so that the plate has radius 1) — in the form

$$\frac{1}{r^2}\left(r\,(r\,T_r)_r + T_{\theta\theta} \right) = \Delta T = 0, \quad \text{for } 0 \le \theta \le 2\pi \quad \text{and } 0 \le r \le 1, \tag{4.1}$$

with the boundary condition † $T_r(1,\,\theta) = h(\theta)$, for some given function h. (4.2)
† T and h are 2π-periodic in θ.

In addition, the function $h = h(\theta)$ must satisfy $$\int_0^{2\pi} h(\theta)\, d\theta = 0. \tag{4.3}$$

Equation (4.3) follows because, unless the net heat flux through the boundary vanishes, no steady state temperature is possible. Equivalently: *(4.1–4.2) does not have a solution unless (4.3) applies.*

The solution to this problem, as follows from separation of variables, is [4]

$$T = \sum_{-\infty<n<\infty \quad n\neq 0} \frac{1}{|n|}\, h_n\, r^{|n|}\, e^{i\,n\,\theta}, \qquad \text{where} \quad h_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} h(\theta)\, e^{-i\,n\,\theta} d\theta, \tag{4.4}$$

where, **to make the solution unique,**[5] **we have added the condition: $T = 0$ at the origin.**

For $r < 1$ this formula provides a solution to the equation even if h is not very smooth (or in fact, not even a function) and thus has a very poorly convergent Fourier series $\sum h_n\, e^{i\,n\,\theta}$. For example, assume that the sequence $\{h_n\}$ is bounded. Then, in any disk of radius $R < 1$: (i) the series for T in (4.4) converges absolutely; and (ii) the series obtained by term by term differentiation, for any derivative of T, also converges absolutely. It follows that T is smooth for $r < 1$. In a similar fashion, one can justify exchanging the order of integration and summation — as you are asked to do below.

1. Assume that $h = \delta(\theta) - \dfrac{1}{2\,\pi}$, and **write an explicit formula for the solution T.**
2. In (4.4), substitute the formula for h_n into the series for T, exchange the summation and integration order, and
   derive an equation of the form $$T = \int_0^{2\,\pi} G(r,\,\theta-\phi)\, h(\phi)\, d\phi. \tag{4.5}$$
3. Let $H = G_r + \dfrac{1}{2\,\pi}$. **Show that:**
   (a) $H(r,\,\theta) \to 0$ as $r \to 1$, if $\theta \neq 0$. (b) $\displaystyle\int_0^{2\,\pi} H(r,\,\theta)\, d\theta = 1$ for $r < 1$.

Hint. Add the positive and negative indexes n separately. Then use $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n}\, z^n = -\ln(1-z)}$ *for* $|z| < 1$ *— note that, for* $|z| < 1$ *the argument for the log stays within a circle of radius 1 centered at 1, so that no problem arises because of the branch point of the log at the origin.*

4.2 Answer: Laplace equation in a circle #02

Let us first **assume that $\boldsymbol{h = \delta(\theta) - \frac{1}{2\,\pi}}$.** Then $h_n = \dfrac{1}{2\,\pi}$ for all $n \neq 0$, so that

$$2\,\pi\, T = \sum_{n=1}^{\infty} \frac{1}{n}\, r^n\, e^{i\,n\,\theta} + \sum_{n=1}^{\infty} \frac{1}{n}\, r^n\, e^{-i\,n\,\theta} = -\ln\left((1-r\,e^{i\,\theta})(1-r\,e^{-i\,\theta})\right).$$

Thus

$$T = -\frac{1}{2\,\pi}\, \ln(1 - 2\,r\cos\theta + r^2) = G(r,\,\theta), \tag{4.6}$$

where G is defined by the equation — note that $1 - 2\,r\cos\theta + r^2 \geq (1-r)^2$, so that there is no problem with the meaning of ln is the formula.

More generally, $h_n = \dfrac{1}{2\,\pi}\displaystyle\int_0^{2\,\pi} h(\phi)\, e^{-i\,n\,\phi}\, d\phi$ leads to

$$2\,\pi\, T = \int_0^{2\,\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n}\, r^n\, e^{i\,n\,(\theta-\phi)} + \sum_{n=1}^{\infty} \frac{1}{n}\, r^n\, e^{-i\,n\,(\theta-\phi)}\right) h(\phi)\, d\phi.$$

[4] Check, by direct substitution, that this is indeed the solution. That (4.2) is satisfied should be "obvious".
[5] A constant can be added to any solution of (4.1–4.2).

Thus the general formula

$$T = \int_0^{2\pi} G(r,\, \theta - \phi)\, h(\phi)\, d\phi, \tag{4.7}$$

where G is as in (4.6).

Finally, it is easy to check that $$H = G_r + \frac{1}{2\pi} = \frac{1}{2\pi}\,\frac{(1-r)(1+2\cos\theta - r)}{1 - 2\,r\cos\theta + r^2}.$$
From this it follows that

(a) $1 - 2\,r\cos\theta + r^2 = (r - \cos\theta)^2 + 1 - \cos^2\theta > 1 - \cos^2\theta$. Thus $\lim_{r\to 1} H(r,\,\theta) = 0$ if $\theta \neq 0$.

(b) For $r < 1$, $H = \frac{1}{2\pi}\left(\sum_{n=1}^{\infty} r^{n-1}\, e^{i\,n\,\theta} + \sum_{n=1}^{\infty} r^{n-1}\, e^{-i\,n\,\theta} + 1\right)$, which converges absolutely, and can be integrated term by term. Thus $\int_0^{2\pi} H(r,\,\theta)\, d\theta = 1$ for $r < 1$.

The same argument in (b) shows that, for $r < 1$, $$\int_0^{2\pi} H(r,\,\theta)\, h(\theta)\, d\theta = h_0 + \sum_1^{\infty} r^{n-1}\,(h_n + h_{-n}).$$
If h is smooth enough $\boldsymbol{h(0) = \sum_{-\infty}^{\infty} h_n}$, where convergence is absolute. Thus $\boldsymbol{H \to \delta(\theta)}$ **as** $\boldsymbol{r \to 1}$**.**

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5 Normal modes #02

5.1 Statement: Normal modes #02

Normal mode expansions are not guaranteed to work when the associated eigenvalue problem is not self adjoint (more generally, not normal).[6] Here is an example: consider the problem

$$u_t + u_x = 0 \quad \text{for}\ \ 0 < x < 1 \quad \text{and} \quad t > 0, \qquad \text{with}\ \ u(x,\,0) = U(x) \tag{5.1}$$

and boundary condition $\boldsymbol{u(0,\,t) = 0}$. **Find all the normal mode solutions** to this problem,[7] and **show that the solution to (5.1) cannot be written as a linear combination of these modes. What is the solution to (5.1)?**

5.2 Answer: Normal modes #02

Substitute $u = e^{\lambda\,t}\,\phi(x)$ into (5.1). This yields the eigenvalue problem

$$\lambda\,\phi + \phi' = 0, \quad \text{for}\ \ 0 < x < 1, \quad \text{with}\ \phi(0) = 0, \tag{5.2}$$

where a prime denotes a derivative. Hence $\phi \equiv 0$ for any λ. Thus **there are no normal modes!**
The solution to (5.1) is:

$$u = f(t - x), \tag{5.3}$$

where $f(\zeta) = U(-\zeta)$ for $-1 < \zeta < 0$, and $f(\zeta) = 0$ for $\zeta > 0$. Note that the solution vanishes for $t > 1$, while a normal mode would (at best) decay exponentially [another way to see that there can be no normal modes].

THE END.

[6] An operator A is normal if it commutes with its adjoint: $A\,A^\dagger = A^\dagger\,A$.
[7] That is: non-trivial solutions of the form $u = e^{\lambda\,t}\,\phi(x)$, where λ is a constant.