Answers to Problem Set #03, 18.306
MIT (Winter-Spring 2021)

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1.1 Statement: Information loss and traffic flow shocks

In this problem we will show that shocks in traffic flow are associated with information loss, and will derive a formula that quantifies this loss. For simplicity, consider periodic solutions to the traffic flow equation

$$\rho_t + q_x = 0, \quad \text{periodic of period } T > 0, \tag{1.1}$$

where the traffic flux is given by $q = Q(\rho)$. Assume that there is a single shock per period, at $x = \sigma(t)$. The shock velocity, s, is given by the Rankine-Hugoniot jump condition

$$\frac{d\sigma}{dt} = s = \frac{q_R - q_L}{\rho_R - \rho_L}, \tag{1.2}$$

where ρ_R (resp.: ρ_L) is the value of ρ immediately to the right (resp.: the left) of the shock discontinuity, $q_R = Q(\rho_R)$, and $q_L = Q(\rho_L)$. In addition, the shock satisfies the *entropy condition*

$$c_L > s > c_R, \tag{1.3}$$

where $c = \dfrac{dQ}{d\rho}$ is the characteristic speed, $c_R = c(\rho_R)$, and $c_L = c(\rho_L)$. Now, let

$$\mathcal{I} = \mathcal{I}(t) = \int_{\text{period}} \frac{1}{2}\,\rho^2(x,\,t)\,dx. \tag{1.4}$$

1. **Show that:**
$$\frac{d\mathcal{I}}{dt} = \frac{1}{2}\,(q_L - q_R)\,(\rho_L + \rho_R) + \int_{\rho_L}^{\rho_R} \rho\,c(\rho)\,d\rho. \tag{1.5}$$
Hint. For any short enough time period, a constant "a" exists such that $a < \sigma(t) < a + T$ during the time interval. Given this, write $\mathcal{I}$ in the form $\boldsymbol{\mathcal{I} = \int_a^\sigma \frac{1}{2}\,\rho^2\,dx + \int_\sigma^{a+T} \frac{1}{2}\,\rho^2\,dx}$, before taking the time derivative. Then use the fact that, in each of the two intervals ρ satisfies
$$\rho_t = -c(\rho)\,\rho_x = -\frac{1}{\rho}\,f_x \quad \textit{where} \quad f = \int_0^\rho s\,c(s)\,ds. \tag{1.6}$$
In addition, use (1.2) to eliminate $\dot{\boldsymbol{\sigma}}$ from the resulting formulas.

2. **Show that:**
$$\frac{d\mathcal{I}}{dt} < 0 \tag{1.7}$$
by interpreting the right hand side in (1.5) geometrically, as (the negative of) the area between two curves — such area known to be positive. *Hint. Use that $\boldsymbol{\rho\,c = \frac{d}{d\rho}(\rho\,q) - q}$, and (1.9) below.*

Remark 1.1 *Recall that, in traffic flow, $q = Q(\rho)$ is a concave function,*[1] *vanishing at both $\rho = 0$ and the jamming density $\rho_j > 0$, positive for $0 < \rho < \rho_j$, with a maximum (the road capacity q_m) at some value $0 < \rho_m < \rho_j$. Because of this,* **the conditions (1.2–1.3) are equivalent to:**[2]

The shock velocity is given by the slope of the secant line joining $(\rho_L,\,q_L)$ to $(\rho_R,\,q_R)$. Further $\rho_L < \rho_R$. (1.8)

Note that **For $\rho_L < \rho < \rho_R$, the curve $q = Q(\rho)$ is *above* the secant line,** *(1.9)*
because Q is concave.

Remark 1.2 *How does $\mathcal{I}$ in (1.4) related to "information"? Basically, we argue that a function contains the least amount of information when it is a constant, and that the more "oscillations" it has, the more information it carries. A (rough) measure of this is given by*

$$\mathcal{J} = \int_{\text{period}} \frac{1}{2}\,(\rho - \overline{\rho})^2\,dx, \tag{1.10}$$

where $\overline{\rho}$ is the average value of ρ (note that conservation guarantees that $\overline{\rho}$ is a constant in time). Alternatively, write ρ the Fourier series $\rho = \sum_n \rho_n(t)\,e^{i\,n\,2\,\pi\,x/T}$ for ρ. Then the amount of "oscillation" in ρ can be characterized by $\sum_{n\neq 0} \frac{1}{2}\,|\rho_n|^2$, which is the same as (1.10). It is easy to see that

$$\mathcal{J} = \mathcal{I} - T\,\overline{\rho}^2. \tag{1.11}$$

Hence $\dot{\mathcal{J}} = \dot{\mathcal{I}}$, so that (1.7) indicates that the shock is driving ρ towards a constant value.

The notion of "information" that we use here is not the same as that in "information theory" (which requires a stochastic process of some kind). Neither is it the same as the related notion of "entropy" of statistical mechanics (for this we would need a statistical theory[3] *for traffic flow). Nevertheless, $S = -\mathcal{J}$ plays the same role for (1.1)*

[1] Concave means that $\frac{d^2Q}{d\rho^2} < 0$. Note also that **only solutions satisfying $0 \le \rho \le \rho_j$ are acceptable.**
[2] Thus shocks move slower than cars, and the density a driver sees increases as the car goes through a shock.
[3] Such theories exist.

that entropy plays for gas dynamics.

Finally, we point out that it can be shown that: for any convex function $h = h(\rho)$ (i.e.: $h'' > 0$)

$$\frac{d}{dt}\int_{\text{period}} h(\rho)\,dx < 0 \quad \textit{when shocks are present,}$$

though showing this is a bit more complicated than for the special case $h = \frac{1}{2}\rho^2$.

1.2 Answer: Information loss and traffic flow shocks

To derive (1.5), we follow the hint and write $\mathcal{I} = \int_a^\sigma \frac{1}{2}\rho^2\,dx + \int_\sigma^{a+T}\frac{1}{2}\rho^2\,dx$. This yields

$$\begin{aligned}
\dot{\mathcal{I}} &= \frac{1}{2}\,\dot\sigma\,(\rho_L^2-\rho_R^2) + \int_a^\sigma \rho\,\rho_t\,dx + \int_\sigma^{a+T}\rho\,\rho_t\,dx \\
&= \frac{1}{2}\,(q_L-q_R)(\rho_L+\rho_R) - \int_a^\sigma f_x\,dx - \int_\sigma^{a+T} f_x\,dx \\
&= \frac{1}{2}\,(q_L-q_R)(\rho_L+\rho_R) - f_L + f_a - f_{a+T} + f_R \\
&= \frac{1}{2}\,(q_L-q_R)(\rho_L+\rho_R) + \int_{\rho_L}^{\rho_R}\rho\,c(\rho)\,d\rho, \qquad (1.12)
\end{aligned}$$

where we have used (1.2), the fact that the solution is periodic (thus $f_a = f_{a+T}$), and the definition of f (thus $f_R - f_L = \int_{\rho_L}^{\rho_R}\rho\,c(\rho)\,d\rho$). Now we use the fact that $\rho\,c = \frac{d}{d\rho}(\rho\,q) - q$ to write

$$\begin{aligned}
\dot{\mathcal{I}} &= \frac{1}{2}\,(q_L-q_R)(\rho_L+\rho_R) + \rho_R\,q_R - \rho_L\,q_L - \int_{\rho_L}^{\rho_R} Q(\rho)\,d\rho, \\
&= \frac{1}{2}\,(q_L+q_R)(\rho_R-\rho_L) - \int_{\rho_L}^{\rho_R} Q(\rho)\,d\rho. \qquad (1.13)
\end{aligned}$$

On the other hand, the secant line joining $(\rho_L,\, q_L)$ to $(\rho_R,\, q_R)$ is given by

$$y = q_R\frac{\rho-\rho_L}{\rho_R-\rho_L} + q_L\frac{\rho_R-\rho}{\rho_R-\rho_L} \quad\Longrightarrow\quad \int_{\rho_L}^{\rho_R} y\,d\rho = \frac{1}{2}\,(q_L+q_R)(\rho_R-\rho_L) \qquad (1.14)$$

It follows that
$$\dot{\mathcal{I}} = -A < 0, \qquad (1.15)$$
where ***A* is the area** between the curve $q = Q(\rho)$ and the secant line joining $(\rho_L,\, q_L)$ to $(\rho_R,\, q_R)$, in the segment $\rho_L < \rho < \rho_R$. That $A > 0$ follows from (1.9). See figure 1.1.

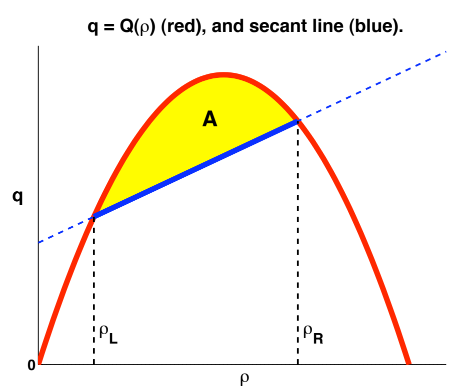

Figure 1.1: The picture on the right illustrates the area A, determined by the (concave) traffic flow function $q = Q(\rho)$ and the secant line joining $(\rho_L,\, q_L = Q(\rho_L))$ to $(\rho_R,\, q_R = Q(\rho_R))$. The negative of this area is the rate at which $\mathcal{I}$, defined in (1.4), decreases.

2 IVP for a kinematic wave equation #01

2.1 Statement: IVP for a kinematic wave equation #01

Consider the Kinematic Wave equation

$$u_t + q_x = 0, \qquad \text{where} \qquad q = \frac{1}{2}\,u^2, \tag{2.1}$$

and u is the density for some conserved quantity. Using the method of characteristics, the Rankine-Hugoniot jump conditions, and the entropy conditions, find the solutions (for **all** $t > 0$) to the following initial value problems:

1st Initial Value Problem:

$$u(x,\,0) = \begin{cases} 1 & \text{for} \quad -\infty < x \leq -1, \\ -x & \text{for} \quad -1 \leq x \leq 0, \\ 0 & \text{for} \quad 0 \leq x < \infty. \end{cases} \tag{2.2}$$

2nd Initial Value Problem:

$$u(x,\,0) = \begin{cases} 0 & \text{for} \quad -\infty < x \leq -1, \\ 1+x & \text{for} \quad -1 \leq x \leq 0, \\ 1-x & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{for} \quad 1 \leq x < \infty. \end{cases} \tag{2.3}$$

2.2 Answer: IVP for a kinematic wave equation #01

The characteristics for the equation are $\dfrac{dx}{dt} = u$, along which $\dfrac{du}{dt} = 0$ — hence they are straight lines along which u = constant = slope of line. In order to solve the problems we proceed as follows:

- **1st.** Find the characteristic starting from *every* point where data is given.
- **2nd.** Find the regions were multiple values occur (more than one characteristic going through them) and resolve the multiple values by introducing shocks.

HINT: as you follow the solutions below, have scratch paper on hand, and *draw* the characteristics, shocks, etc. in space-time. This is very helpful to develop understanding of how all this works.

1st Initial Value Problem:

1. The characteristics starting on $-\infty < x = \zeta_1 \leq -1$ and $t = 0$ are given by: $x = t + \zeta_1$, along which $u = 1$.
2. The characteristics starting on $-1 \leq x = \zeta_2 \leq 0$ and $t = 0$ are given by: $x = -\zeta_2\, t + \zeta_2$, along which $u = -\zeta_2 = \dfrac{x}{t-1}$.
3. The characteristics starting on $0 \leq x = \zeta_3 < \infty$ and $t = 0$ are given by: $x = \zeta_3$, along which $u = 0$.
4. For $0 < t < 1$ there is no intersection of characteristics, and the solution is given by

$$u(x,\,t) = \begin{cases} 1 & \text{for} \quad -\infty < x \leq t-1, \\ \dfrac{x}{t-1} & \text{for} \quad t-1 \leq x \leq 0, \\ 0 & \text{for} \quad 0 \leq x < \infty. \end{cases} \tag{2.4}$$

5. At $t = 1$ the characteristics starting on $-1 \le x = \zeta_2 \le 0$ and $t = 0$ converge on the point $x = 0$. For $t > 1$ the region $0 \le x \le t - 1$ has multiple characteristics through every point. In order to resolve this problem a shock must be introduced starting at $x = 0$ and $t = 1$, such that the characteristics cease to exist when they reach the shock. This shock then connects the region where $u = 1$ (behind the shock), with the region where $u = 0$ (ahead of the shock). Hence, from the Rankine-Hugoniot conditions, it has speed $s = \dfrac{[q]}{[u]} = \dfrac{1}{2}$. The solution for $t \ge 1$ is then given by

$$u(x,\,t) = \begin{cases} 1 & \text{for} \quad -\infty < x < \frac{1}{2}(t-1), \\ 0 & \text{for} \quad \frac{1}{2}(t-1) < x < \infty. \end{cases} \tag{2.5}$$

Notice that **the entropy condition is satisfied by the shock**, as the characteristics converge on it from the left and from the right. **YOU SHOULD VERIFY THIS.**

2nd Initial Value Problem:

1. The characteristics starting on $-\infty < x = \zeta_1 \le -1$ and $t = 0$ are given by:
$x = \zeta_1$, along which $u = 0$.

2. The characteristics starting on $-1 \le x = \zeta_2 \le 0$ and $t = 0$ are given by:
$x = (1 + \zeta_2)\,t + \zeta_2$, along which $u = 1 + \zeta_2 = \dfrac{x+1}{t+1}$.

3. The characteristics starting on $0 \le x = \zeta_3 \le 1$ and $t = 0$ are given by:
$x = (1 - \zeta_3)\,t + \zeta_3$, along which $u = 1 - \zeta_3 = \dfrac{x-1}{t-1}$.

4. The characteristics starting on $1 \le x = \zeta_4 < \infty$ and $t = 0$ are given by:
$x = \zeta_4$, along which $u = 0$.

5. For $0 < t < 1$ there is no intersection of characteristics, and the solution is given by

$$u(x,\,t) = \begin{cases} 0 & \text{for} \quad -\infty < x \le -1, \\ (x+1)/(t+1) & \text{for} \quad -1 \le x \le t, \\ (x-1)/(t-1) & \text{for} \quad t \le x \le 1, \\ 0 & \text{for} \quad 0 \le x < \infty. \end{cases} \tag{2.6}$$

6. At $t = 1$ the characteristics starting on $0 \le x = \zeta_3 \le 1$ and $t = 0$ converge on the point $x = 1$. For $t > 1$ the region $1 \le x \le t$ has multiple characteristics through every point. To resolve this problem we introduce a shock starting at $x = 1$ and $t = 1$, such that the characteristics cease to exist when they reach the shock. This shock connects the region where $u = (x + 1)(t + 1)$ (behind the shock), with the region where $u = 0$ (ahead of the shock). The Rankine-Hugoniot conditions yield the following O.D.E. for the shock position $x = x_s(t)$

$$\frac{dx_s}{dt} = \frac{[q]}{[u]} = \frac{1}{2}\,\frac{x_s+1}{t+1}, \tag{2.7}$$

where $x_s(1) = 1$. Thus $\boldsymbol{x_s = -1 + \sqrt{2\,(t+1)}}$ and the solution for $t \ge 1$ is given by

$$u(x,\,t) = \begin{cases} 0 & \text{for} \quad -\infty < x \le -1, \\ (x+1)/(t+1) & \text{for} \quad -1 \le x < x_s, \\ 0 & \text{for} \quad x_s < x < \infty. \end{cases} \tag{2.8}$$

The entropy condition is satisfied by the shock, as the characteristics converge on it from the left and from the right. **YOU SHOULD VERIFY THIS.**

7. Notice that the value of the solution right behind the shock is given by $u_s = \sqrt{\dfrac{2}{t+1}}$ — hence the shock strength decreases like the inverse of the square root of time.

3 Smooth kinematic waves have infinite conservation laws

3.1 Statement: Smooth kinematic waves have infinite conservation laws

Consider the (kinematic wave) equation for some conserved scalar density $\rho = \rho(x,\, t)$ in one dimension

$$\rho_t + q_x = 0, \quad \text{where} \quad q = Q(\rho) \tag{3.1}$$

is the flow function — which we assume is sufficiently nice (say, it has a continuous first derivative). **SHOW THAT:** if $f = f(\rho)$ is some arbitrary (sufficiently nice) function, then there exists a function $g = g(\rho)$ such that *for any classical*[4] *solution of (3.1), we have*

$$f_t + g_x = 0. \tag{3.2}$$

In other words, (formally) f behaves as a "conserved" quantity, with flow function g. In particular:

$$\frac{d}{dt}\int_a^b f\,dx = g\big|_{x=a} - g\big|_{x=b} \tag{3.3}$$

expresses the "conservation" of f for any interval $a < x < b$.

Very important: it is *crucial that the solution ρ have derivatives in the classical sense.* **When shocks are present, this result is false.** As you will be asked to show in another problem, when shocks are present (3.2) has to be replaced by

$$f_t + g_x = \sum_{\text{shocks}} c_s\,\delta(x - x_s(t)), \tag{3.4}$$

where $x = x_s(t)$ are the shock positions, δ is the Dirac delta function, and the $c_s = c_s(t)$ are some coefficients that are *generally not zero.* Thus the shocks act as sources (or sinks) for f.

3.2 Answer: Smooth kinematic waves have infinite conservation laws

Let ρ be a classical solution of equation (3.1), and consider equation (3.2), which can be written in the form

$$f'(\rho)\,\rho_t + g'(\rho)\,\rho_x = 0. \tag{3.5}$$

Since $\rho_t + c(\rho)\,\rho_x = 0$ — where $\boldsymbol{c = dQ/d\rho}$ — it should be clear that (3.5) will be satisfied provided that $g' = c\,f'$. In other words, **the answer to this problem is**

$$g(\rho) = \int^{\rho} c(s)\,f'(s)\,ds. \tag{3.6}$$

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4 The zero viscosity limit for scalar convex conservation laws

4.1 Statement: The zero viscosity limit for scalar convex conservation laws

Consider a scalar convex conservation law, with a small amount of "viscosity" added. Namely:

$$\rho_t + q_x = \nu\,\rho_{x\,x}, \quad \text{with} \quad q = q(\rho) \quad \text{smooth and convex:} \quad \frac{d^2 q}{d\rho^2} \geq C_q > 0, \quad \text{and} \quad \rho(x,\,0) = f(x), \tag{4.1}$$

[4] That is: no shocks, so no generalized derivatives are involved.

where C_q is some constant, $0 < \nu \ll 1$, f is smooth, and f and all its derivatives vanish (rapidly) as $|x| \to \infty$. We want to investigate the behavior, as $\nu \to 0$, of the solution to this problem. In particular, **we want to compute**

$$\mathcal{I} = \mathcal{I}(t) = \int_{-\infty}^{\infty} \Psi(\rho(x,\, t))\, dx, \quad \text{as } \nu \to 0, \tag{4.2}$$

where Ψ is a smooth convex function [5] such that $\Psi(0) = 0$. Furthermore, let $h = h(\rho)$ be defined by

$$\frac{dh}{d\rho} = c(\rho)\, \frac{d\Psi}{d\rho} \text{ and } h(0) = 0, \quad \text{where} \quad c = \frac{dq}{d\rho}. \tag{4.3}$$

Note 1. We will assume that the solution $\rho = \rho(x,\, t)$ to (4.1) exists, that it is smooth, that ρ and all its derivatives vanish (rapidly) as $|x| \to \infty$, and that ρ is unique (within this class).

Note 2. Here we will use concepts introduced in the problem Lax Entropy condition for scalar convex conservation laws and information loss — you should read the statement for this exercise. This will also serve the purpose of giving meaning to $\mathcal{I}$ as a measure of the "wavy-ness" of the solution.

Multiplying by $\dfrac{d\Psi}{d\rho}$ the equation $\rho_t + c(\rho)\, \rho_x = \nu\, \rho_{x\,x}$ satisfied by ρ, and using (4.3), we obtain

$$\Psi(\rho)_t + h(\rho)_x = \nu\, \Psi'(\rho)\, \rho_{x\,x}, \tag{4.4}$$

where the prime denotes differentiation with respect to ρ. Integrating this equation then yields

$$\frac{d\mathcal{I}}{dt} = \nu \int_{-\infty}^{\infty} \Psi'(\rho)\, \rho_{x\,x}\, dx = -\nu \int_{-\infty}^{\infty} \Psi''(\rho)\, \rho_x^2\, dx < 0, \tag{4.5}$$

which shows that $\mathcal{I}$ is, always, a decreasing function of time.

From (4.5) is seems natural to conclude that: in the limit $\nu \to 0$, $\mathcal{I}$ becomes constant. However, **this is false.** As $\nu \to 0$, the solution to (4.1) develops thin transition layers (shocks) where the derivatives become large, so that the right hand side in (4.5) does not vanish as $\nu \to 0$. **Your task here is to check this fact.** *Proceed as follows.*

First, consider the case where the solution to (4.1) develops a single shock as $\nu \to 0$, along some path $x = \sigma(t)$. Then,

> For $0 < \nu \ll 1$, the solution near $x = \sigma$ (for any fixed time t) can be described by a traveling wave solution to the equation in (4.1), of the form
>
> $$\rho \sim \mathcal{R}(z), \quad \text{where} \quad z = \frac{x - s\,t}{\nu} \quad \text{and} \quad s = \frac{d\,\sigma}{d\,t},$$
>
> with limits ρ_R as $z \to \infty$ and ρ_L as $z \to -\infty$. On the other hand, away from $x = \sigma$, the derivatives of ρ remain bounded as $\nu \to 0$. (4.6)

Use (4.6) and (4.5) to **compute the $\nu \to 0$ limit** of $\dfrac{d\mathcal{I}}{dt}$. You should be able to **show that**

$$\frac{d\mathcal{I}}{dt} \quad \longrightarrow \quad h_R - h_L - s\,(\Psi_R - \Psi_L), \tag{4.7}$$

where the subscript R indicates evaluation at ρ_R, and the subscript L indicates evaluation at ρ_L. *Notice that this is the same formula for the contribution to the rate of change of $\mathcal{I}$ by a shock* listed in the statement for the problem Lax Entropy condition for scalar convex conservation laws and information loss — see the 9^{th} equation there.

[5] Hence $\dfrac{d^2\,\Psi}{d\rho^2} \geq C_\Psi > 0$, for some constant C_Ψ.

Hint. Consider the first integral in (4.5). Away from $x = \sigma$, the derivatives of the solution remain bounded, and the contribution of these regions to the integral vanishes as $\nu \to 0$. Hence we can limit the integration to a small region near the shock, namely

$$\frac{d\mathcal{I}}{dt} \approx \nu \int_{\sigma-\epsilon}^{\sigma+\epsilon} \Psi'(\rho)\, \rho_{x\,x}\, dx, \quad 0 < \nu \ll \epsilon \ll 1, \tag{4.8}$$

where the traveling wave approximation $\rho \sim \mathcal{R}(z)$ in (4.6) can be used. From this, using the o.d.e. that $\mathcal{R}$ satisfies,[6] the result in (4.7) follows.

Second, if the solution to (4.1) develops more than one shock as $\nu \to 0$, then a calculation as the one above can be done near each of the shock locations $x = \sigma_\ell(t)$, obtaining as the result that the right hand side in (4.7) is replaced by a sum including the contributions for each one of the shocks.

Remark 4.1 *The result in this problem illustrates a persistent phenomena in nonlinear p.d.e. that incorporate "small" perturbations involving the highest derivatives in the equation. Then, as the perturbations vanish, their effects on the solution* **do not.** *For example: in high Reynolds number flows thin boundary layers can form (mostly near walls), where viscous effects dominate, and contribute a finite amount (that does not go away as the Reynolds number goes to infinity) to the flow behavior. In other cases high frequency (thin) structures appear over large regions of the solution, which again do not go away as the perturbations vanish. This second type of situation is quite often associated with open problems where a good mathematical theory is lacking — e.g.: collision-less shocks in plasmas, turbulence, etc.*

4.2 Answer: The zero viscosity limit for scalar convex conservation laws

Following the hint, we write

$$\frac{d\mathcal{I}}{dt} \approx \nu \int_{\sigma-\epsilon}^{\sigma+\epsilon} \Psi'(\rho)\, \rho_{x\,x}\, dx \approx \int_{-\epsilon/\nu}^{+\epsilon/\nu} \Psi'(\mathcal{R})\, \frac{d^2\,\mathcal{R}}{dz^2}\, dz \approx \int_{-\infty}^{+\infty} \Psi'(\mathcal{R})\, \frac{d^2\,\mathcal{R}}{dz^2}\, dz, \tag{4.9}$$

where $0 < \nu \ll \epsilon \ll 1$ and $\mathcal{R}$ satisfies

$$\frac{d^2\,\mathcal{R}}{dz^2} = \Big(c(\mathcal{R}) - s\Big)\, \frac{d\mathcal{R}}{dz}. \tag{4.10}$$

Hence, taking the limit $\nu \to 0$, we can write

$$\frac{d\mathcal{I}}{dt} = \int_{-\infty}^{+\infty} \Psi'(\mathcal{R})\, \frac{d^2\,\mathcal{R}}{dz^2}\, dz = \int_{-\infty}^{+\infty} \Psi'(\mathcal{R})\, \Big(c(\mathcal{R}) - s\Big)\, \frac{d\mathcal{R}}{dz}\, dz = \int_{-\infty}^{+\infty} \frac{d}{dz}\, \Big(h(\mathcal{R}) - s\,\mathcal{R}\Big)\, dz. \tag{4.11}$$

From this (4.7) follows, since $\mathcal{R} \to \rho_R$ as $z \to +\infty$, and $\mathcal{R} \to \rho_L$ as $z \to -\infty$.

5 Traveling waves for the KdV-Burgers equation

5.1 Statement: Traveling waves for the KdV-Burgers equation

Consider the p.d.e. (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\, u_{x\,x} + \mu\, u_{x\,x\,x}, \quad \text{where } \nu > 0 \text{ and } \mu \neq 0 \text{ are both small.} \tag{5.1}$$

[6] Note that **you do not need to have an explicit formula for** $\mathcal{R}$. Knowing the o.d.e. that $\mathcal{R}$ satisfies, and knowing the limits as $z \to \pm\infty$ for $\mathcal{R}(z)$, **should be enough.**

This p.d.e. has been proposed as a simple model for the structure of bores in shallow water. [7]

Your task is: Study the behavior of the physically meaningful traveling wave solutions of this equation, as a function of the equation parameters ν, μ, and any wave parameters (e.g.: wave amplitude, speed, etc.). Are there solutions that approach constants as $x \pm \infty$? If so, how are the constants approached (monotone, oscillations)? Are the constants equal, or different? Are there periodic, oscillatory solutions? Make plots sketching the types of solutions that are possible.

Note: to be **physically meaningful**, a solution has to be **bounded.**

HINT. Exploit problem symmetries to reduce the number of free parameters: Because the equation is Galilean invariant, you need only look at time independent solutions (zero propagation speed):

$$\left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{x\,x} + \mu\,u_{x\,x\,x}, \tag{5.2}$$

This can be integrated to

$$\mu\,u_{x\,x} = -\nu\,u_x + \frac{1}{2}\,u^2 + \kappa, \tag{5.3}$$

where κ is a constant. By a re-scaling of the dependent and independent variables, this last equation can be reduced to one of the following three cases, each having a single parameter

$$u_{x\,x} = -\delta\,u_x + \frac{1}{2}\,u^2 + \sigma, \quad \text{where } \sigma = \pm 1 \text{ or } \sigma = 0, \text{ and } 0 < \delta < \infty. \tag{5.4}$$

Study the physically relevant solutions for these equations, as δ varies.

Remark 5.1 *There is no explicit solution for (5.4). One approach is to write (5.4) as a phase plane system, and find the critical points and their type, the null-clines, etc. Then use this information to deduce the phase portrait for the equation. From this the answer to the problem should follow.*

5.2 Answer: Traveling waves for the KdV-Burgers equation

Equation (5.1) is *Galilean invariant*: if $u = u(x,\,t)$ solves (5.1), then $\tilde{u} = u(x - g,\,t) + \dot{g}$ is also a solution — for any function $g = g(t)$. Hence, when studying traveling wave solutions, it is enough to look at the zero velocity case — namely, to equation (5.2), which can be integrated to (5.3).

Introduce

$$\boldsymbol{u = \alpha\,\tilde{u}(\tilde{x}),} \quad \text{where} \quad \boldsymbol{\tilde{x} = \beta\,x,} \quad \text{and} \tag{5.5}$$

- **-1-** For $\boldsymbol{\kappa \neq 0}$: $\boldsymbol{\alpha = \text{sign}(\mu)\,|\kappa|}$ and $\boldsymbol{\beta = \text{sign}(\mu)\,\sqrt{|\kappa|/|\mu|}}$.
- **-2-** For $\boldsymbol{\kappa = 0}$: $\boldsymbol{\alpha = \nu^2/\mu}$ and $\boldsymbol{\beta = \nu/\mu}$.

Then equation (5.3) takes the form [8] in (5.4) where

- **-3-** For $\boldsymbol{\kappa \neq 0}$: $\boldsymbol{\delta = \nu/\sqrt{|\kappa\,\mu|}}$ and $\boldsymbol{\sigma = \text{sign}(\kappa/\mu)}$.
- **-4-** For $\boldsymbol{\kappa = 0}$: $\boldsymbol{\delta = 1}$ and $\boldsymbol{\sigma = 0}$.

From now on we work with the equation in the form (5.4), hence there should not be any confusion produced by dropping the tildes in the variables, to simplify the notation.

Multiply (5.4) by $2\,u_x$. Then

$$\left(u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma\,u\right)_x = -2\,\delta\,u_x^2. \tag{5.6}$$

Hence

$$\textbf{Equation (5.4) has NO nontrivial periodic traveling waves.} \tag{5.7}$$

[7] See §13.15 of Whitham's book: *Linear and nonlinear waves.*

[8] **We drop the tildes to simplify the notation.**

Proof: If u is a periodic solution of (5.4), then (5.6) yields $\int_{\text{period}} u_x^2\,dx = 0$. Hence u is constant.

From (5.6), $\boldsymbol{\psi = u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma\,u}$ is non-increasing. Thus the limits $\boldsymbol{\psi(\pm\infty) = \lim_{x\to\pm\infty} \psi(x)}$ must exist, where the limits need not be finite. In the **case where u is a physical solution**

$$\infty > \psi(-\infty) > \psi(\infty) > -\infty, \quad \textbf{if the solution is nontrivial.} \tag{5.8}$$

In particular **both u and u_x are bounded.**

Proof: Since u is bounded, $\psi(\pm\infty) = -\infty$ is impossible. Since ψ is non-increasing, $\psi(\infty) = \infty$ is also impossible — as it would imply $\psi \equiv \infty$. Hence we need only examine the case $\psi(-\infty) = \infty$. But, since u is bounded, this can only happen if either $u_x \to \infty$ or $u_x \to -\infty$, which would contradict the fact that u is bounded. Finally $\psi(-\infty) = \psi(\infty)$ is possible only if $u_x \equiv 0$, which yields u constant.

Furthermore: **when u is a physical solution**

$$\boldsymbol{u_x \to 0} \ \textbf{and} \ \boldsymbol{u \to u_*} \ \textbf{as} \ \boldsymbol{x \to \pm\infty}, \quad \textbf{where} \ \boldsymbol{u_*} \ \textbf{is a constant.} \tag{5.9}$$

To show this notice that (5.6) yields, for $\boldsymbol{\psi = u_x^2 - \frac{1}{3}\,u^3 - 2\,\sigma}$ and $\boldsymbol{u}$, the system

$$\psi_x = -2\,\delta\,(\psi - \Psi(u)) \quad \text{and} \quad u_x = \pm\sqrt{\psi - \Psi(u)}, \ \text{ where } \quad \Psi(u) = -\frac{1}{3}\,u^3 - 2\,\sigma\,u. \tag{5.10}$$

This applies for $\boldsymbol{\psi \geq \Psi(u)}$, and the sign of the square root can switch ONLY along $\boldsymbol{\psi = \Psi(u)}$.

Remark 5.2 *What happens with the solutions to (5.10) along $\psi = \Psi(u)$ follows from (5.4). From the definition of ψ, $\psi = \Psi(u)$ is equivalent to $u_x = 0$. But there (5.4) yields $u_{xx} = \frac{1}{2}\,u^2 + \sigma$, hence u_x flips sign —* **except** *if $\frac{1}{2}\,u^2 + \sigma = 0$ — and the solution continues. Note that:*

r2a. *If a solution satisfies (somewhere) $u_x = 0$ and $u = u_*$ — where $\frac{1}{2}\,u_*^2 + \sigma = 0$ — then $u = u_*$ everywhere. This follows from uniqueness of the initial value problem for (5.4).*

r2b. *The points where $\frac{1}{2}\,u^2 + \sigma = 0$ are the extrema of Ψ, i.e.: where $\Psi'(u) = 0$.*

Proof of (5.9): From remark **5.2**, what the **solutions to (5.10)** do in the $(u,\,\psi)$-plane follows — see figure 5.1. **As x increases:** ψ decreases, with u either increasing or decreasing.[9] This continues till the solution either reaches the curve $\psi = \Psi(u)$, or it diverges — with $\psi \to -\infty$ (non-physical solution). When/if the solution reaches $\psi = \Psi(u)$, if this happens at a point where $\Psi'(u) \neq 0$, the solution "bounces" of the curve $\psi = \Psi(u)$ — with u_x flipping sign, and the process continues. A non-trivial solution can reach $\psi = \Psi(u)$ at a point where $\Psi'(u) = 0$ only as $x \to \infty$ — otherwise it would be identically constant, as follows from **r2a**. In fact, a non-trivial physical solution has to approach such a point as $x \to \infty$, for the alternative is $\psi \to -\infty$ (non-physical). **An entirely similar argument applies for x decreasing.**

The proof above not only justifies (5.9), it also shows that

$$\textbf{The constant } \boldsymbol{u_*} \textbf{ in (5.9) must satisfy } \boldsymbol{u_*^2 + 2\,\sigma = 0.} \tag{5.11}$$

[9] Depending on which sign for the square root is selected.

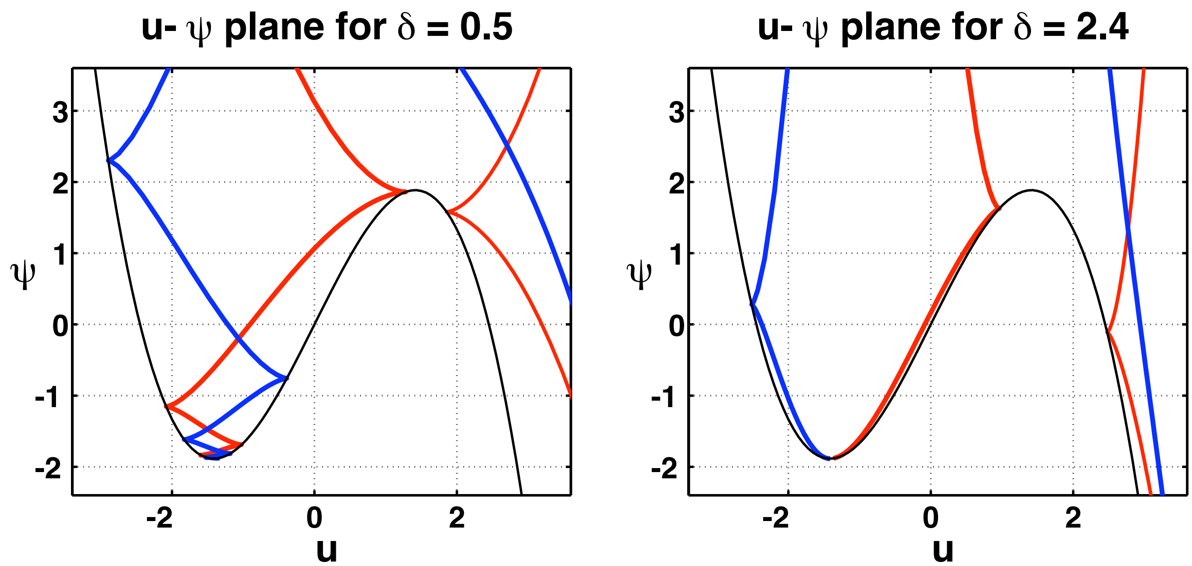

Figure 5.1: Solutions to equation (5.4), for $\sigma = -1$, as they behave in the $(\psi,\, u)$-plane — see (5.10). On the left panel δ is small, and one can see the solutions bouncing back and forth inside the dip in the $\psi = -(1/3)\,u^3 + u$ curve. On the right panel, δ is larger, and the solutions tend to stay close to the curve $\psi = -(1/3)\,u^3 + u$. This is easy to see from the equation for ψ in (5.10): if δ is large, ψ decays rapidly towards $\psi = -(1/3)\,u^3 + u$. The critical value for δ at which the behavior of the solutions switches is $\delta_c = \sqrt{4\sqrt{2}} \approx 2.38$.

Hence: **There is no physical solution if $\sigma = 1$.** (5.12)

Furthermore: **There is no non-trivial physical solution if $\sigma = 0$.** (5.13)

This second result follows because in this case there is only one u_*, so that (5.8) cannot be satisfied.

Case $\sigma = -1$.

In this case there are two possible values for u_*, given by $u_* = \sqrt{2}$ and $u_* = -\sqrt{2}$. From the prior arguments, **if there is a physical solution, it must connect $u = \sqrt{2}$ (with $u_x = 0$) at $x = -\infty$ to $u = -\sqrt{2}$ (with $u_x = 0$) at $x = \infty$. Such a solution exists, and is unique (up to translations).**

Existence: The curve $\psi = \Psi(u)$ has a local minimum at $u = -\sqrt{2}$, with $\Psi = -4\sqrt{2}/3$, and a local maximum at $u = \sqrt{2}$, with $\Psi = 4\sqrt{2}/3$. Consider a solution to (5.10), starting at $x = -\infty$ from the local maximum, with the negative square root sign — one of the curves on the left panel of figure 5.1 is very close to such a solution. It is easy to see, using arguments like those after remark **5.2**, that this solution is trapped inside the bowl shaped region above the local minimum of Ψ. Hence, it must reach the local minimum as $x \to \infty$. This argument proves existence, except for a little detail that must be clarified: is there such a thing as *"a solution for (5.10) starting at $x = -\infty$, from the local maximum, with …"?* Below we show that the answer is yes, by identifying these solutions with the unique orbit leaving a saddle point in the direction of decreasing u.

Uniqueness: Equation (5.4), for $\sigma = -1$, can be written as the phase plane system

$$u_x = v \quad \text{and} \quad v_x = -\delta\, v + \frac{1}{2}\, u^2 - 1, \tag{5.14}$$

This system has two critical points: $C_1 = (u,\, v) = (\sqrt{2},\, 0)$ and $C_2 = (u,\, v) = (-\sqrt{2},\, 0)$ — which correspond to the local maximum and minimum of Ψ. It is easy to see that C_1 is a saddle point, so that there is a single orbit leaving

C_1 towards $u < \sqrt{2}$. In terms of ψ and u, the solutions corresponding to this orbit are the ones that the existence argument uses. Since the orbit is unique, there is a unique (up to translation) traveling wave solution. **Note:** That this saddle orbit ends up at C_2 as $x \to \infty$ is not easy to show using (5.14). This is why we introduced (5.10).

Finally: What is the nature of the critical point C_2? It is easy to see that

S1. If $\delta > \delta_c = 2^{5/4}$, C_2 is a node. In this case the traveling wave solution connecting $u = \sqrt{2}$ at $x = -\infty$ to $u = -\sqrt{2}$ at $x = \infty$ looks like a standard shock connection: a smooth, monotone decreasing, function connecting the states at $\pm\infty$. See figure 5.2.

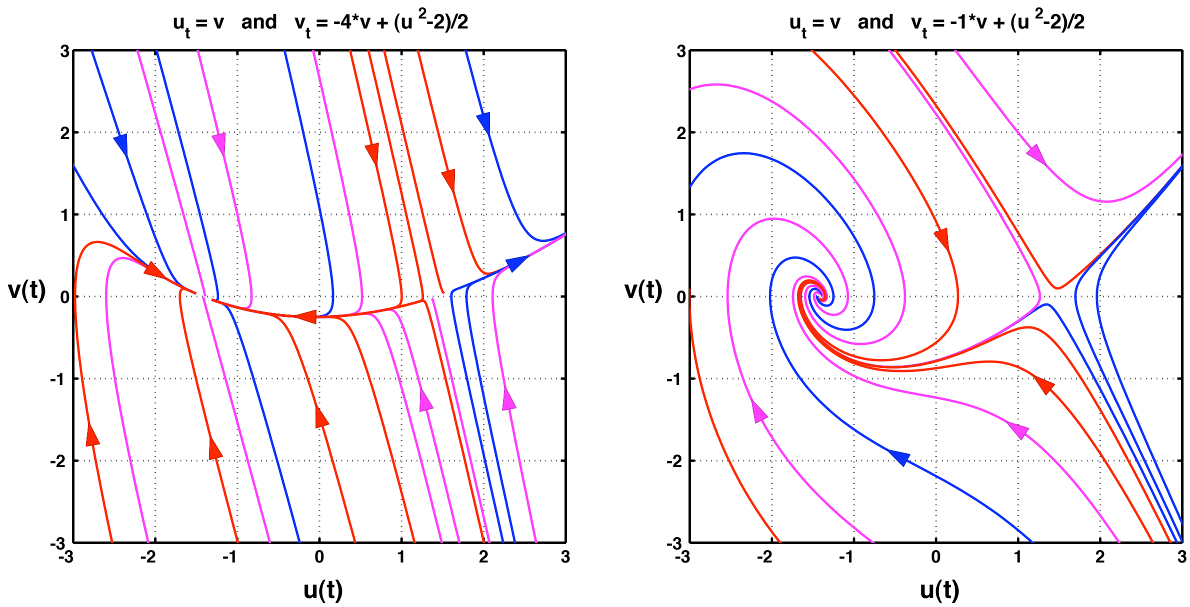

Figure 5.2: Phase plane portrait for (5.14). Left panel: super-critical case $\delta > \delta_c$. Traveling wave is a saddle-node connection. Right panel: sub-critical case $\delta < \delta_c$. Traveling wave oscillates as $x \to \infty$.

S2. If $\delta < \delta_c = 2^{5/4}$, C_2 is a spiral point. In this case the traveling wave solution connecting $u = \sqrt{2}$ at $x = -\infty$ to $u = -\sqrt{2}$ at $x = \infty$ oscillates as it approaches its limit at ∞. Instead of a standard shock connection, this is now an **undulating shock, or bore.**
Note that, to know which way the oscillations occur in the "physical" variables, you must transform these results using (5.5). See figure 5.2.

Remark 5.3 *Many of the arguments in this answer could have been made much shorter using well known facts from the theory of phase plane systems. Specifically: Poincaré-Bendixon theory and Index theory. I decided against using them, since such theories are not a normal part of the pre-requisite courses for 18.306. You can learn it if you take 18.385.*

Summary/Conclusions

We have shown that, **for every $0 < \delta < \infty$ there exists a unique $U = U(z,\, \delta)$ defined by:** [10]

u1. $\lim_{z\to-\infty} U(z,\,\delta) = \sqrt{2}$, and $\lim_{z\to\infty} U(z,\,\delta) = -\sqrt{2}$.

u2. $U(z,\,\delta) > 0$ for $z < 0$ and $U(0,\,\delta) = 0$.

u3. $U'' = -\delta\, U' + \frac{1}{2}U^2 - 1.$ where the **primes** indicate derivatives with respect to z.

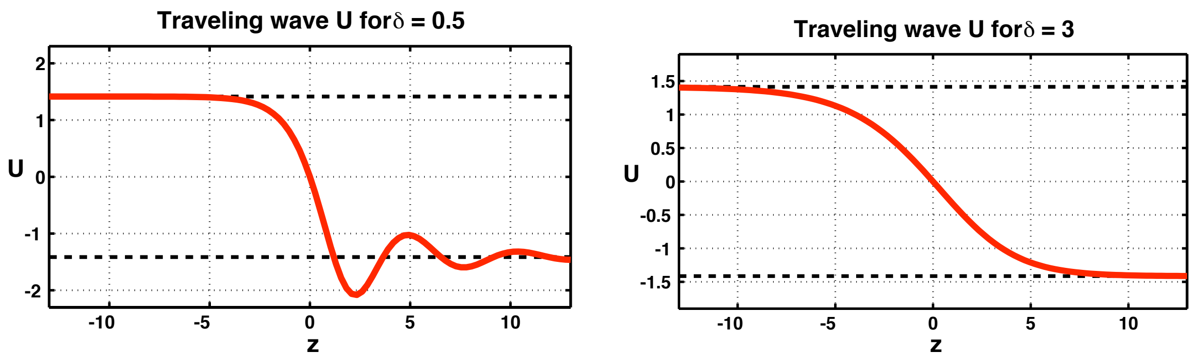

Figure 5.3: Example profiles for $U = U(z,\,\delta)$, as a function of z, for two values of δ. On the left panel, where $\delta = 0.5 < \delta_c$, the solution oscillates as $z \to \infty$. On the right panel, where $\delta = 3 > \delta_c$, the solution is monotone.

See figure 5.3 for typical plots for U.

Let $\boldsymbol{\delta_c = \sqrt{4\,\sqrt{2}}}$. Then

u4. As $\boldsymbol{z \to -\infty}$, $\boldsymbol{U}$ **approaches** $\boldsymbol{\sqrt{2}}$ **exponentially** .... $\boldsymbol{U \sim \sqrt{2} - e^{\lambda_1\,(z - z_1)}}$,
where z_1 is a constant, and $\boldsymbol{\lambda_1 = \frac{1}{2}\left(\sqrt{\delta^2 + \delta_c^2} - \delta\right) > 0}$.

u5. If $\boldsymbol{\delta > \delta_c}$, as $\boldsymbol{z \to \infty}$, $\boldsymbol{U}$ **approaches** $\boldsymbol{-\sqrt{2}}$ **exponentially** .... $\boldsymbol{U \sim -\sqrt{2} + e^{-\lambda_2\,(z - z_2)}}$,
where z_2 is a constant, and $\boldsymbol{\lambda_2 = \frac{1}{2}\left(\delta - \sqrt{\delta^2 - \delta_c^2}\right) > 0}$.

u6. If $\boldsymbol{\delta < \delta_c}$, as $\boldsymbol{z \to \infty}$, $\boldsymbol{U}$ **approaches the limit** $\boldsymbol{-\sqrt{2}}$
with exponentially
damped oscillations .... $\boldsymbol{U \sim -\sqrt{2} + e^{-\delta\,(z - z_3)/2}\cos\left(\frac{1}{2}\sqrt{\delta_c^2 - \delta^2}\,(z - z_4)\right)}$,
where z_3 and z_4 are some constants.

Then **all the physically meaningful traveling waves for (5.1) are given by**

$$\boldsymbol{u = \alpha\,U\,(\beta\,(x - s\,t,\;\delta)) + s} \tag{5.15}$$

where, in terms of the limits of the traveling wave as $\boldsymbol{x \to \pm\infty}$, namely $\boldsymbol{u_- > u_+}$, we have

$$\boldsymbol{s = \frac{u_- + u_+}{2}}, \quad \boldsymbol{\alpha = \mathrm{sign}(\mu)\,\frac{[u]}{2\,\sqrt{2}}}, \quad \text{and} \quad \boldsymbol{\beta = \mathrm{sign}(\mu)\,\sqrt{\frac{[u]}{2\,\sqrt{2}\,|\mu|}}}, \tag{5.16}$$

with $\boldsymbol{[u] = u_- - u_+}$ and $\boldsymbol{\delta = \frac{\nu}{\sqrt{2\,|\mu|\,[u]}}\,\delta_c}$.

THE END.

[10] Condition **u2** eliminates the translational degree of freedom in traveling waves, resulting in a unique answer.