Answers to Problem Set #02, 18.306
MIT (Winter-Spring 2021)

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1 Compute a channel flow rate function #01

1.1 Statement: Compute a channel flow rate function #01

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{1.1}$$

where $A = A(x, t)$ is the cross-sectional filled area of the river bed, x measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate q should be a function of A only[1] follows from the assumption of quasi-equilibrium. Then q is determined by a local balance between the friction forces and the force of gravity down the river bed.

Assume now a man-made channel, with uniform triangular cross-section† and a uniform (small) downward slope, characterized by an angle θ. Assume also that the frictional forces are proportional to the product of the flow velocity u down the channel, and the wetted perimeter P_w of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function Q should have.** † Isosceles triangle, with bottom angle ϕ.

Hints: (1) $Q = u\,A$, *where* u *is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter* P_w *is proportional to some power of* A.

1.2 Answer: Compute a channel flow rate function #01

The wetted perimeter is proportional to the square root of the filled cross-sectional area. That is:[2] $P_w = C_w \sqrt{A}$, where $\boldsymbol{C_w = \sqrt{8/\sin(\phi)}}$ and ϕ is the *bottom angle of the triangular channel bed.* Thus the frictional forces (per unit length) along the river bed are given by $F_f = C_f\, P_w\, u = C_f\, C_w\, u\, \sqrt{A}$ — where u is the flow velocity and C_f is a *friction coefficient.*

On the other hand, the component of the force of gravity (per unit length) along the channel bed is given by $F_g = \rho\, g\, \sin(\theta)\, A$ — where θ is the *angle that the channel bed makes with the horizontal,* g is the *acceleration of gravity,* and ρ is the *density of the water* in the channel.

From the quasi-equilibrium assumption $F_f = F_g$. This yields $u = \dfrac{\rho\, g\, \sin(\theta)}{C_f\, C_w}\sqrt{A}$. Hence, since $q = u\,A$, it follows that:

$$\boldsymbol{Q = \frac{\rho\, g\, \sin(\theta)}{C_f\, C_w}\, A^{\frac{3}{2}}.} \tag{1.2}$$

[1] Possibly also x. That is: $q = Q(x,\, A)$, to account for non-uniformities along the river.
[2] Write the height and base of the water-filled-triangle in terms of the half bottom angle and the wetted side $= P_w/2$.

Note that q is a convex function of A.

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2 Gas flow through a porous media

2.1 Statement: Gas flow through a porous media

Consider the situation where a gas is flowing through a porous media. If conditions are not varying too rapidly,[3] the gas flow velocity through the media is determined by the balance of the pressure force driving the flow, and the viscous resistance to the flow through the pores. This balance is subsumed by *Darcy's law*

$$\vec{u} = -\tfrac{\kappa}{\mu}\,\nabla\, p, \tag{2.1}$$

where: $\vec{u}$ is the gas flow velocity, κ is the *permeability* of the porous media, μ is the *dynamic viscosity of the gas*, ∇ is the gradient operator, and p is the pressure.
Units: $[\kappa] = \text{length}^2$, $[\mu] = \frac{\text{mass}}{\text{length}\times\text{time}}$, while $[p] = \frac{\text{force}}{\text{area}}$.

Consider now *a 1D flow of an ideal gas.*

— For example: flow down a pipe filled with sand, where the pipe has constant cross-section.

— For an ideal gas $p = R\,T\,\rho$, where R = gas constant, T = temperature, and ρ = gas density.

QUESTIONS:

1. Assume that *the flow occurs at constant temperature*, and **derive an equation for $\boldsymbol{\rho = \rho(x,\,t)}$.**

2. If the flow is adiabatic, instead of isothermal, $p = \beta\,\rho^\gamma$, where β is a constant and γ is the ratio of specific heats for the gas. **What is the equation for $\boldsymbol{\rho = \rho(x,\,t)}$ in this case?**

2.2 Answer: Gas flow through a porous media

Mass is conserved, thus it should be

$$\rho_t + q_x = 0, \tag{2.2}$$

where q is the mass flow. However, $q = \rho\, u$, and u is given by (2.1), so that

$$u = -\tfrac{\kappa}{\mu}\, p_x. \tag{2.3}$$

Furthermore, in both cases the pressure is a function of the density only, $p = p(\rho)$. Thus we get the equation

$$\rho_t = \left(D(\rho)\,\rho_x\right)_x, \quad \text{where} \quad D = \frac{\kappa}{\mu}\,\rho\,\frac{dp}{d\rho}. \tag{2.4}$$

For the isothermal flow

$$D = \frac{\kappa\, R\, T}{\mu}\,\rho. \tag{2.5}$$

For the adiabatic flow

$$D = \frac{\kappa\,\beta\,\gamma}{\mu}\,\rho^\gamma. \tag{2.6}$$

Equation (2.4) is a nonlinear diffusion equation, where D is the *diffusivity* — D has units of length square over time. Furthermore, since $\rho \geq 0$ and $\frac{dp}{d\rho} > 0$, $D \geq 0$. However, when ρ vanishes, so does D. This leads to *sharp diffusion fronts,* unlike those produced by linear diffusion (where there is no clear edge to a diffusing blob).

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[3] The standard "quasi-equilibrium" assumption used when deriving continuum equations from conservation laws.

3 Linear 1st order PDE (problem 01)

3.1 Statement: Linear 1st order PDE (problem 01)

Part 1. Find the general solutions to the two 1st order linear scalar PDE

$$x\,u_x + y\,u_y = 0, \qquad \text{and} \qquad y\,v_x - x\,v_y = 0. \tag{3.1}$$

Hint: *The general solutions take a particular simple form in polar coordinates.*

Part 2. For u, find the solution such that on the circle $x^2 + y^2 = 2$, it satisfies $u = x$. *Where is this solution determined by the data given?*

Part 3. Is there a solution to the equation for v such that $v(x, 0) = x$, for $-\infty < x < \infty$?

Part 4. How does the general solution for u changes if the equation is modified to

$$x\,u_x + y\,u_y = (x^2 + y^2)\,\sin(x^2 + y^2)? \tag{3.2}$$

3.2 Answer: Linear 1st order PDE (problem 01)

Part 1. In terms of polar coordinates $x = r\,\cos(\theta)$ and $y = r\,\sin(\theta)$, the equations take the form

$$r\,u_r = 0 \quad \text{and} \quad -v_\theta = 0\,. \tag{3.3}$$

Hence, the general solutions are $u = U(\theta)$ and $v = V(r)$, where U and V are arbitrary functions of their arguments.

Alternatively, the characteristics for the first equation (namely $\boldsymbol{\frac{dx}{ds} = x}$ and $\boldsymbol{\frac{dy}{ds} = y}$, along which $\boldsymbol{\frac{du}{ds} = 0}$) are the straight lines through the origin. Hence u should only be a function of the angle. Similarly, the characteristics for the second equation are the circles centered at the origin, so that v should depend on the radial variable only.

Remark 3.1 *Notice that the origin $x = y = 0$ is a "bad" point for these p.d.e.'s — where no restriction is applied on the solutions. Hence, we generally should exclude it from the domain where we seek to solve the equations.*

Part 2. Since for $x^2 + y^2 = 2$ we have $x = \sqrt{2}\,\cos(\theta)$ and $y = \sqrt{2}\,\sin(\theta)$, from the general solution in part 1 it follows that

$$u = \sqrt{2}\,\cos(\theta)\,. \tag{3.4}$$

This solution is determined for $r > 0$ — see remark **3.1**. In fact, it is singular and becomes multiple valued for $r \to 0$.

Part 3. From part 1, it should be that $v(x,\,0) = V(|x|) = v(-x,\,0)$, which the data $v(x,\,0) = x$ does not satisfy. Hence the problem in part 3 does not have a solution. Another way of seeing this is that the characteristics for this problem (circles centered at the origin) intersect the x-coordinate axis at two points each, where the data given is not consistent.

Part 4. The characteristic curves are still the same, namely $\dfrac{dx}{ds} = x$ and $\dfrac{dy}{ds} = y$, but now along then u is no longer constant and

$$\frac{du}{ds} = (x^2 + y^2)\,\sin(x^2 + y^2) = -\frac{1}{2}\,\frac{d}{ds}\cos(x^2 + y^2). \tag{3.5}$$

Hence, the general solution is $u = U(\theta) - \dfrac{1}{2}\,\cos(r^2) =$ "general solution of the homogeneous problem" + "particular solution" — since the problem is linear.

Alternatively: in polar coordinates the equation is $r\,u_r = r^2\,\sin(r^2)$, which leads to the same answer.

4 Linear 1st order PDE (problem 03)

4.1 Statement: Linear 1st order PDE (problem 03)

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{4.1}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to find the solution. Write, **explicitly**, $\boldsymbol{u = u(x,\,y)}$ as a function of x and y, using f.

Hint. Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$ (where $-\infty < \tau < \infty$). Finally: eliminate τ, to get u as a function of x and y.

Part 2. In which part of the $(x,\,y)$ plane is the solution determined?

Hint. Draw in the x-y plane the characteristics computed in part 1.

Part 3. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

4.2 Answer: Linear 1st order PDE (problem 03)

Using x to parametrize the characteristic equations for the problem in (4.1), we obtain

$$dy/dx = 2\,x \quad \text{and} \quad du/dx = y, \tag{4.2}$$

with $y = \tau$ and $u = f(\tau)$ for $x = 0$ and $-\infty < \tau < \infty$. These equations have the solution

$$y = \tau + x^2 \quad \text{and} \quad u = f(\tau) + \tau\,x + \frac{1}{3}\,x^3. \tag{4.3}$$

Part 1. The first expression in (4.3) yields $\tau = y - x^2$. Then, from the second expression

$$u = f(y - x^2) + x\,(y - x^2) + \frac{1}{3}\,x^3. \tag{4.4}$$

Part 2. The characteristic curves, as given by (4.3), are parabolas intersecting the y-axis at $y = \tau$ — all shifted versions of the parabola $y = x^2$. Hence they cover the whole plane, with exactly one characteristic through every point. Thus **the solution is uniquely defined over the whole plane.**

Part 3. From (4.4) it should be clear that: *if f has a continuous derivative, then the partial derivatives of u are also continuous.*

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5 Linear 1st order PDE (problem 10)

5.1 Statement: Linear 1st order PDE (problem 10)

Integrating factors. Show that the pde

$$(a(x,\,y)\,\mu)_y = (b(x,\,y)\,\mu)_x \tag{5.1}$$

is a necessary and sufficient condition guaranteeing that $\mu = \mu(x,\,y) \neq 0$ is an integrating factor for the ode

$$a(x,\,y)\,dx + b(x,\,y)\,dy = 0, \tag{5.2}$$

in any open subset of the plane without holes.

Part II. Assume that $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$.
Find an integrating factor for (5.2) — i.e.: obtain a nontrivial solution of (5.1). **Use it to integrate** (5.2), and **write (5.2) in the form**

$$\Phi(x,\, y) = \text{constant}, \quad \text{for some function } \Phi. \tag{5.3}$$

Hint 5.1 *Solving by characteristics (5.1) leads to (5.2), or equivalent (as part of the process — <u>check this</u>!) To* get out of this circular situation, note that: *for a and b as above, $\mu = F(x,\, y)$ solves (5.1) iff $\mu = F(y,\, x)$ does. This suggests that you should look for solutions*[4] *invariant under this symmetry; that is: $\mu(x,\, y) = \mu(y,\, x)$. Hence* write $\boldsymbol{\mu = \mu(u,\, v)}$, with $\boldsymbol{u = x + y}$ and $\boldsymbol{v = x\,y}$, *since solutions that satisfy $\mu(x,\, y) = \mu(y,\, x)$ must have this form — see remark **5.1**.*

Remark 5.1 *The transformation $(x,\, y) \to (u,\, v)$ is not one to one: it maps the whole xy-plane into the region $v \leq \frac{1}{4}\,u^2$ of the uv-plane, with double valued inverse $x = \frac{1}{2}\,\left(u \pm \sqrt{u^2 - 4\,v}\right)$ and $y = \frac{1}{2}\,\left(u \mp \sqrt{u^2 - 4\,v}\right)$. Furthermore:* **(a)** *The two inverses are related by the $x \leftrightarrow y$ switch.* **(b)** *The singular line $u^2 = 4\,v$ corresponds to the line $x = y$.* **(c)** *The map is a bijection between the regions $x \leq y$ and $4\,v \leq u^2$.* **(d)** *The map is a bijection between the regions $x \geq y$ and $4\,v \leq u^2$.*
From **(c - d)** *we see that: for* any *$\mu = \mu(x,\, y)$, $\mu = f(u,\, v)$ for $x \leq y$ and $\mu = g(u,\, v)$ for $x \geq y$, for some f and g. Then,* if $\boldsymbol{\mu(x,\, y) = \mu(y,\, x)}$, *$f = g$, so that* $\boldsymbol{\mu}$ has the form $\boldsymbol{\mu = \mu(u,\, v)}$.

Part III. **Why is it that this approach cannot be generalized to three variables?** That is, to find integrating factors for equations of the form

$$a(x,\, y,\, z)\, dx + b(x,\, y,\, z)\, dy + c(x,\, y,\, z)\, dz = 0. \tag{5.4}$$

5.2 Answer: Linear 1st order PDE (problem 10)

If (5.1) is satisfied in an open subset without holes, then there exists a function $\Phi = \Phi(x,\, y)$ such that

$$\Phi_x = a\,\mu \quad \text{and} \quad \Phi_y = b\,\mu. \tag{5.5}$$

In fact, pick a fixed point in the set, say Q. Then define

$$\Phi(x,\, y) = \int_\Gamma \mu\, a\, dx + \mu\, b\, dy, \tag{5.6}$$

where Γ is any curve (in the set) connecting Q to $(x,\, y)$. The value of this integral does not depend on the curve Γ, as follows from Green's theorem and (5.1) — i.e.: $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ for any closed curve Λ in the set.[5] Hence (5.6) does define a function — which (obviously) satisfies (5.5).

Given (5.5), equation (5.2) yields

$$0 = \mu\, a\, dx + \mu\, b\, dy = \Phi_x\, dx + \Phi_y\, dy = d\Phi \quad \Longleftrightarrow \quad \Phi = \text{constant}. \tag{5.7}$$

Of course, for (5.7) to be of any use, μ must be non-trivial: $\mu = 0$ always works, but it also leads to the useless function $\Phi(x,\, y) \equiv \text{constant}$.

Vice-versa, if μ is an integrating factor, there is some function Φ such that

$$\mu\, a\, dx + \mu\, b\, dy = d\Phi = \Phi_x\, dx + \Phi_y\, dy. \tag{5.8}$$

[4] Note that you only need **one** nontrivial solution.
[5] **Here is where having a set without holes matters:** if there are holes, then $\oint_\Lambda \mu\, a\, dx + \mu\, b\, dy = 0$ is not guaranteed, and (5.6) cannot be used to define a function.

Hence (5.5) applies, from which (5.1) follows.

Part II. With $a = 3\,x\,y + 2\,y^2$, and $b = 3\,x\,y + 2\,x^2$, equation (5.1) takes the form

$$a\,\mu_y - b\,\mu_x = (b_x - a_y)\,\mu = (x - y)\,\mu. \tag{5.9}$$

In terms of the coordinates $u = x + y$ and $v = x\,y$, a little bit of algebra reduces this equation to

$$(y - x)\,(2\,u\,\mu_u - v\,\mu_v + \mu) = 0. \tag{5.10}$$

That is, **for $x \neq y$ we have**

$$2\,u\,\mu_u - v\,\mu_v + \mu = 0. \tag{5.11}$$

We **only need one non-trivial solution for this equation to find an integrating factor.** Nevertheless, next we find **all** the solutions, using characteristics.

The characteristic equations for (5.11) can be written in the form

$$\frac{du}{2\,u} = -\frac{dv}{v} = -\frac{d\mu}{\mu} = ds, \tag{5.12}$$

where s is a parameter along each characteristic. From the first equality it follows that $u\,v^2 = \zeta$, where ζ is a constant on each characteristic — which we use as a label for the characteristic curve. From the second equality it follows that $\mu/v = f$, where f is also a constant along each characteristic — hence $f = f(\zeta)$ must be some function of the characteristic label. Thus the general solution to (5.11) has the form

$$\mu = v\,f(u\,v^2) = x\,y\,f\left((x + y)\,x^2\,y^2\right), \tag{5.13}$$

where f is some arbitrary function (with, at least, one derivative).

Remark 5.2 *The formula in (5.13) follows from solving (5.11), which is equivalent to (5.9) only for $x < y$, or $x > y$. Hence, at this stage, the only thing that we can say about the general solution to (5.9) is that it has the form*

$$\mu = x\,y\,f\left((x + y)\,x^2\,y^2\right) \quad \textit{for } x > y, \quad \textit{and} \quad \mu = x\,y\,g\left((x + y)\,x^2\,y^2\right) \quad \textit{for } x < y,$$

for some arbitrary function f and g. However, evaluating along $x = y$ yields

$$x^2\,f\left(2\,x^5\right) = x^2\,g\left(2\,x^5\right).$$

Thus, we conclude that:

All the solutions to (5.9) satisfy $\mu(x,\,y) = \mu(y,\,x)$, and have the form in (5.13),

with the same function f for all x and y. This is rather interesting, since (generally) the fact that a pde. has a symmetry **does not** imply that all the solutions have it too. *For example: the heat equation $T_t = T_{x\,x}$ is invariant under $x \leftrightarrow -x$, but $T = 2 + \sin(x)\,e^{-t}$ is a solution that is not invariant under $x \leftrightarrow -x$.*

We now take the simplest of the solutions in (5.13), $\mu = v = x\,y$, and plug it into equation (5.5). This yields

$$\Phi_x = a\,x\,y = 3\,x^2\,y^2 + 2\,x\,y^3 \quad \text{and} \quad \Phi_y = b\,x\,y = 3\,x^2\,y^2 + 2\,x^3\,y. \tag{5.14}$$

Thus $\boldsymbol{\Phi = x^3\,y^2 + x^2\,y^3 + \Phi_0}$ — where Φ_0 is a constant. Hence, from (5.7), it follows that: **For the case $\boldsymbol{a = 3\,x\,y + 2\,y^2}$ and $\boldsymbol{b = 3\,x\,y + 2\,x^2}$, (5.2) can be integrated to**

$$x^3\,y^2 + x^2\,y^3 = u\,v^2 = \text{constant}. \tag{5.15}$$

Part III. If μ is a non-trivial integrating factor for (5.4), then

$$\mu\, a\, dx + \mu\, b\, dy + \mu\, c\, dz = d\Phi, \tag{5.16}$$

for some function $\Phi = \Phi(x,\, y,\, z)$. This is equivalent to

$$\mu\, a = \Phi_x, \quad \mu\, b = \Phi_y, \quad \text{and} \quad \mu\, c = \Phi_z. \tag{5.17}$$

However, this then implies the equations

$$(\mu\, a)_y = (\mu\, b)_x, \quad (\mu\, c)_x = (\mu\, a)_z, \quad \text{and} \quad (\mu\, b)_z = (\mu\, c)_y. \tag{5.18}$$

Thus, we get three equations for a single unknown μ. This is an over-determined system that (generally) has only one solution: $\mu = 0$. In fact, in (5.18), multiply the first equation by c, the second equation by b, the third equation by a, add, and use the fact that we want $\mu \neq 0$. This then yields

$$0 = a\,(b_z - c_y) + b\,(c_x - a_z) + c\,(a_y - b_x). \tag{5.19}$$

Equivalently

$$0 = \mathbf{w} \cdot (\nabla \times \mathbf{w})\,, \quad \text{where} \quad \mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{5.20}$$

Hence, if a, b, and c do not satisfy equation (5.20), (5.4) has no integrating factor.

Task left to the reader: *Show that, at least locally, (5.20) is sufficient to guarantee that (5.18) has a non-trivial solution.*

Hint 5.2 *If $\mathbf{w} \equiv 0$, then any μ solves (5.18). Hence, in any sufficiently small cube, we can assume that one of the components of $\mathbf{w}$ is never zero. Thus, without loss off generality, assume $a \geq \delta > 0$ in the cube $-\epsilon < x,\, y,\, z < \epsilon$, where δ and $\epsilon > 0$ are constants. Then:* **I.** *Use the first equation in (5.18) to construct $\mu_0 = \mu_0(x,\, y)$ for $-\epsilon < x,\, y < \epsilon$.* **II.** *Use the second equation in (5.18), with $\mu(x,\, y,\, 0) = \mu_0(x,\, y)$, to define μ in a neighborhood in $\mathcal{R}^{\ni}$ of the square $-\epsilon < x,\, y < \epsilon$.* **III.** *Show that the function μ that you just constructed solves (5.18). To do this:*
Define $\boldsymbol{\phi = (\mu\, a)_y - (\mu\, b)_x}$ and $\boldsymbol{\psi = (\mu\, c)_y - (\mu\, b)_z}$. Then: (i) Use (5.19) to find an algebraic relationship between ϕ and ψ. (ii) Use that μ satisfies the middle equation in (5.18), to derive a p.d.e. that ϕ satisfies. (iii) By construction $\phi = 0$ for $z = 0$. Use the pde in (ii) to conclude that $\phi \equiv 0$. (iv) Use (i) and (iii), conclude that $\psi \equiv 0$. Since μ satisfies the the middle equation in (5.18) by construction, this ends the proof.
Note: point out where the assumption $a > 0$ comes into play in your arguments.

Remark 5.3 *Of course, once a μ satisfying (5.18) is obtained, a Φ satisfying (5.17) is given by*

$$\Phi(x,\, y,\, z) = \int_{\Gamma} \mu\, a\, dx + \mu\, b\, dy, \tag{5.21}$$

where Γ is any curve connecting some fixed point Q with $(x,\, y,\, z)$.
Why is it that the integral in (5.21) depends ONLY on the endpoints of the curve Γ?

6 Semi-Linear 1st order PDE (problem 02)

6.1 Statement: Semi-Linear 1st order PDE (problem 02)

Discuss the problems

$$u_x + 2\,x\,u_y = 2\,x\,u^2, \qquad \text{with} \qquad \begin{cases} \text{(a) } u(x,\,x^2) = 1 & \text{for } \; -1 < x < 1, \\ \text{(b) } u(x,\,x^2) = -\pi/(1+\pi\,x^2) & \text{for } \; -1 < x < 1, \\ \text{(c) } u(x,\,x^2) = (1-x^2/4)^{-1}/4 & \text{for } \; -1 < x < 1. \end{cases} \tag{6.1}$$

How many solutions exist in each case? Where are they uniquely defined?
Note that the data in these problems is prescribed along a characteristic!

6.2 Answer: Semi-Linear 1st order PDE (problem 02)

Using x to parametrize the characteristic equations for the problem in (6.1), we obtain

$$dy/dx = 2\,x \quad \text{and} \quad du/dx = 2\,x\,u^2. \tag{6.2}$$

These equations have the general solution

$$y = \tau + x^2 \quad \text{and} \quad u = \mu\,\left(1 - x^2\,\mu\right)^{-1}. \tag{6.3}$$

where τ and μ are arbitrary constants. For $\tau = 0$ this yields $u(x,\,x^2) = \mu\,\left(1 - x^2\,\mu\right)^{-1}$, which is **inconsistent with (a) in (6.1), consistent with (b) in (6.1)** — provided that $\mu = -\pi$, and **consistent with (c) in (6.1)** — provided that $\mu = 1/4$. Thus

Part 1. Problem (a) in (6.1) has **no solutions.**

Part 2. Problem (b) in (6.1) has **infinitely many solutions.** These are obtained by first using the first expression in (6.3) to obtain $\tau = y - x^2$, and then taking μ as a function of τ in the second expression (with $\mu = -\pi$ for $\tau = 0$). In other words

$$u = f(y - x^2)\,\left(1 - x^2\,f(y - x^2)\right)^{-1}, \tag{6.4}$$

where f is any function such that $f(0) = -\pi$. These solutions are **uniquely determined along the parabola $y = x^2$, for $-\infty < x < \infty$, and nowhere else!**

Part 3. Problem (c) in (6.1) has **infinitely many solutions.** Obtained in the same fashion as for problem (b), these are given by

$$u = f(y - x^2)\,\left(1 - x^2\,f(y - x^2)\right)^{-1}, \tag{6.5}$$

where f is any function such that $f(0) = 1/4$. These solutions are **uniquely determined along the parabola $y = x^2$, for $-2 < x < 2$, and nowhere else!**
Note: The reason that x is restricted to the interval $-2 < x < 2$, is that the solution blows up along the $y = x^2$ characteristic when x reaches ± 2. hence it cannot be (uniquely) extended past these points.

7 Simple finite differences for a 1st order PDE

7.1 Statement: Simple finite differences for a 1st order PDE

Consider the linear PDE

$$u_t + (c(x)\,u)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4}\,\cos(x), \tag{7.1}$$

and $\boldsymbol{u}$ **is periodic of period** $\boldsymbol{2\pi}$ — i.e.: $\boldsymbol{u(x+2\pi,\, t) = u(x,\, t)}$. Assume now that you are asked to calculate the solution of this p.d.e. for $\boldsymbol{0 \le t \le T = 6}$, with **initial condition** given by

$$u(x,\,0) = u_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \le x \le \pi, \tag{7.2}$$

extended periodically outside the interval $[-\pi,\, \pi]$.

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:

$$x_n = -\pi + n\,h \quad \text{for} \quad 1 \le n \le N, \qquad \text{and} \qquad t_m = m\,k \quad \text{for} \quad 0 \le n \le M, \tag{7.3}$$

where N and M are "large" integers, $\boldsymbol{h = 2\pi/N}$, and $\boldsymbol{k = T/M}$. Let $\boldsymbol{u_n^m}$ **be the numerical solution's grid point values.** The expectation is that these values will be related to the exact solution $u(x,\,t)$ by $\boldsymbol{u_n^m = u(x_n,\, t_m) +}$ **small error**, with the error vanishing as N and M grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in h or k. **In all cases the formulas apply for** $\boldsymbol{m \ge 0}$**, with** $\boldsymbol{u_n^0 = u_0(x_n)}$**.**

A. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{h}\,\Big(c(x_n)\,u_n^m - c(x_{n-1})\,u_{n-1}^m\Big)}.$

B. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_n)\,u_n^m\Big)}.$

C. $\displaystyle \boldsymbol{0 = \frac{1}{k}\,(u_n^{m+1} - u_n^m) + \frac{1}{2\,h}\,\Big(c(x_{n+1})\,u_{n+1}^m - c(x_{n-1})\,u_{n-1}^m\Big)}.$

In all cases, once u_n^m is known for some m and all n, u_n^{m+1} can be explicitly computed, for all n. **Note:** when a formula above in **A**, **B** or **C**, calls for a value u_n^m outside the range $1 \le n \le N$, the periodic boundary conditions, which translate into $\boldsymbol{u_{n+N}^m = u_n^m}$, must be used.

The **tasks in this problem are:**

1. **Causality and numerics.** Using arguments based **solely** on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** *(1.1) One of the schemes above cannot possibly work. (1.2) A necessary condition for the other two to work is that a restriction of the form $k \le \lambda\,h$ be imposed on the time step* — where $\lambda > 0$ is a constant that depends on $c = c(x)$.
2. Implement the schemes and try them out with various space resolutions, [6] and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item **1**? **Report what you see, and illustrate it with plots** — a **few** well selected plots should be enough!

7.2 Answer: Simple finite differences for a 1st order PDE

Causality. The key question we have to address first, for both the p.d.e. as well as its discretized versions, is: **what are the domains of dependence?** Namely:

> *Given some arbitrary point $P = (x_0,\, t_0)$ in space-time, what is the region with the property that changes there affect the value of the solution at P?* (7.4)

[6] N in the range $20 \le N \le 200$ should be more than enough to see what happens.

For us here, this is important because **what happens outside the domain of dependence of a point P_0, has NO EFFECT at all on the value of the solution there.** Hence

$$\text{\textit{A necessary condition for the solution computed by a numerical algorithm to converge to the solution of a p.d.e. (as the numerical grid is refined), is: The numerical domain of dependence for any point } P \text{ \textit{should include (as the numerical grid is refined) the p.d.e. domain of dependence for} } P. \tag{7.5}$$

Proof: how can the numerical algorithm get the correct values for the solution of a p.d.e., without using the data that determines the p.d.e. solution? **Note:** restrictions on numerical algorithms that arise in this fashion are called **C.F.L. conditions** (C.F.L. = Courant, Friedrichs, and Lewy).

So, given an arbitrary point $P = (x_0,\, t_0)$, what are the domains of dependence relevant here?

1. **For equation (7.1)** the domain of dependence is given by the characteristic through P. Namely, by the curve determined by the o.d.e. problem:

$$\frac{dx}{dt} = c(x), \quad \text{for } \; t < t_0, \quad \text{with "initial condition" } \; x(t_0) = x_0.$$

   Hence, let $\boldsymbol{c_{\max} = \max_{|x| \leq \pi} c(x) = 1.25,}$ and $\boldsymbol{c_{\min} = \min_{|x| \leq \pi} c(x) = 0.75.}$ Then, **the p.d.e. domain of dependence is included within the wedge:**

$$x_0 + c_{\max}\,(t - t_0) \leq x \leq x_0 + c_{\min}\,(t - t_0), \quad \text{with} \quad t < t_0. \tag{7.6}$$

   Note that, for a "generic" $c = c(x)$, and arbitrary P, this wedge is the best one can do.

2. **For the scheme in item A** the domain of dependence is given by

$$x_0 + \frac{h}{k}\,(t - t_0) \leq x \leq x_0, \quad \text{with} \quad t < t_0. \tag{7.7}$$

   Note that, for any fixed h and k, the domain is a discrete set of points. However, as the grid is refined, the whole (7.7) wedge fills up. The same applies to the wedges in (7.8 – 7.9).

3. **For the scheme in item B** the domain of dependence is given by

$$x_0 \leq x \leq x_0 - \frac{h}{k}\,(t - t_0), \quad \text{with} \quad t < t_0. \tag{7.8}$$

4. **For the scheme in item C** the domain of dependence is given by

$$x_0 + \frac{h}{k}\,(t - t_0) \leq x \leq x_0 - \frac{h}{k}\,(t - t_0), \quad \text{with} \quad t < t_0. \tag{7.9}$$

Thus, **in order for (7.5) to apply, we need:**

— **Scheme in item A.** The wedge in (7.6) will be included within the wedge in (7.7) provided $h/k \geq c_{\max} = 1.25$. Equivalently:

$$k \leq \frac{1}{c_{\max}}\,h = \frac{1}{1.25}\,h \quad \Longleftrightarrow \quad M \geq \frac{1.25}{2\,\pi}\,T\,N. \tag{7.10}$$

— **Scheme in item B.** The wedge in (7.6) **cannot** be included within the wedge in (7.8) for any choice of $h > 0$ and $k > 0$. <u>**This scheme cannot/will not work.**</u>

— **Scheme in item C.** The wedge in (7.6) will be included within the wedge in (7.9) provided $h/k \geq c_{\max} = 1.25$. Equivalently: **(7.10) must apply.**

Condition (7.10) is necessary only. It DOES NOT guarantee that either scheme (A or C) works, nor does it tell us how the scheme in item B fails. To check what happens, I implemented the schemes. The results (which **do not contradict the theoretical conclusions** above) follow.

Implementation of the scheme in item A.

As long as the calculation is performed satisfying the constraint in (7.10) — C.F.L. condition — the scheme appears to perform properly, with convergence as the grid is refined — see figure 7.1.

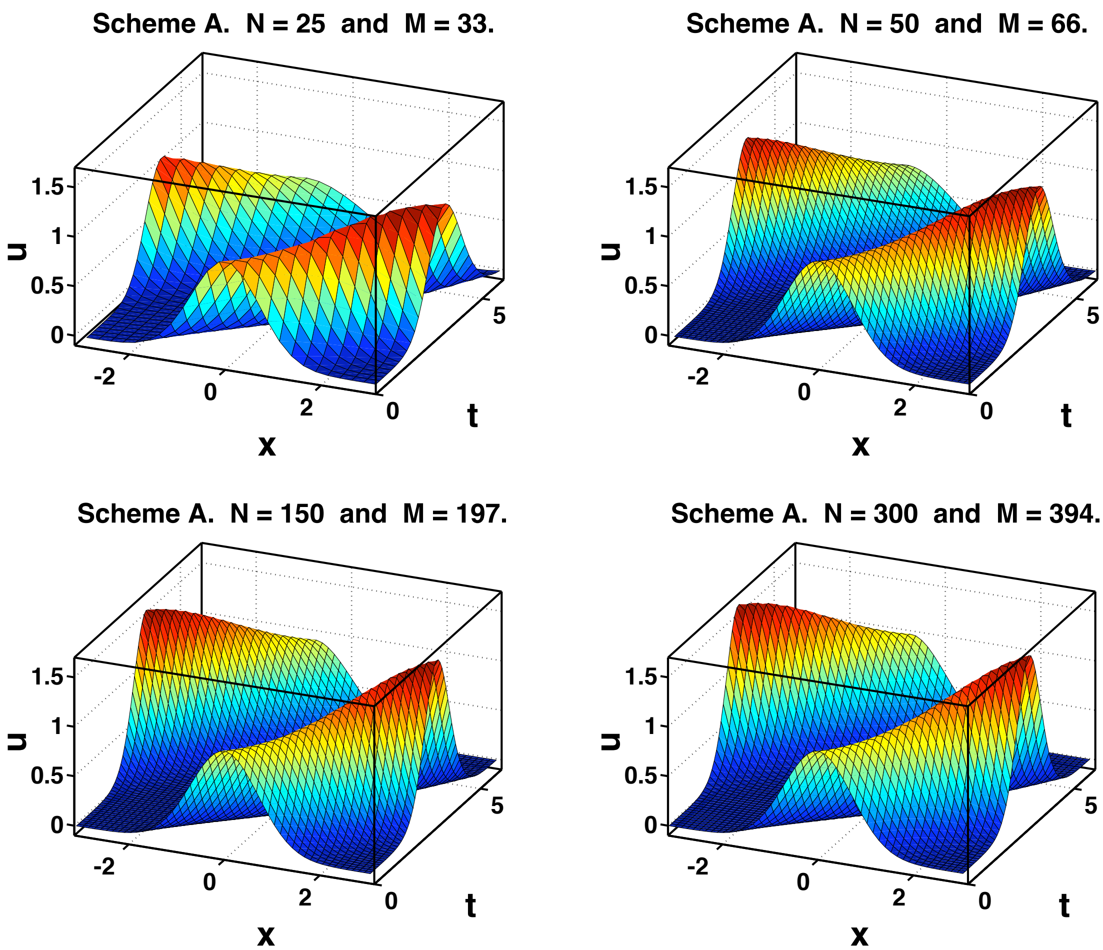


Figure 7.1: Calculations with the scheme in item **A**, to solve (7.1 – 7.2). Left to write, and top to bottom: Numerical solutions as the grid is refined, enforcing the C.F.L. condition (7.10) — specifically, take $M = \text{ceil}\left(1.1\,\dfrac{1.25}{2\,\pi}\,T\,N\right)$. Convergence appears to be occurring.

Remark 7.1 Without an exact solution to compare the answer with, one cannot (solely from numerical calculations) be sure that the observed convergence is to the actual solution — it is not hard to produce reasonable looking numerical schemes that converge to the wrong answer. *But, there exist many tests (using solely numerical results) to increase the confidence level beyond "it looks right".* In the specific case of the scheme in item A, one can prove that: as $h \to 0$, with k restricted by (7.10), the numerical solution converges to the solution to (7.1 – 7.2).

Implementation of the scheme in item B.

Grid scale oscillations (dominant wavelength is $2\,h$), growing exponentially in time (they reach huge amplitudes very quickly) appear. The oscillations' growth rate increases as the grid is refined, with catastrophic results. This scheme is not content with simply failing to work, it fails spectacularly (which is nice: makes it easy to tell it is not working). This is illustrated by the pictures in figure 7.2.

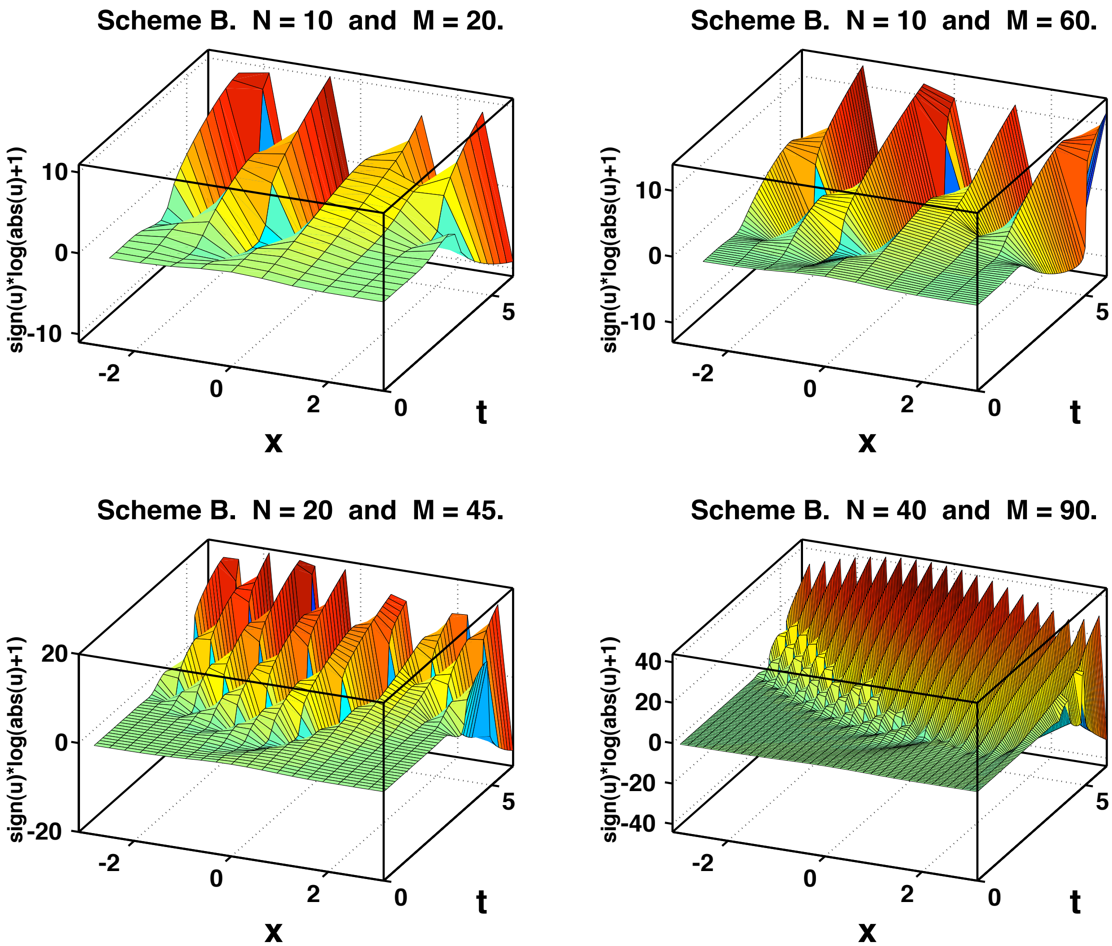


Figure 7.2: Calculations with the scheme in item **B**, to solve (7.1 – 7.2). Left to write, and top to bottom: Numerical solutions, as the grid is refined. M is substantially above the value in (7.10). Larger M's do not help — see top right picture. The spurious oscillations are very large. Hence, we do not plot the numerical solution directly: The plots are for $\mathbf{sign}(\boldsymbol{u})\ \mathbf{ln}(\mathbf{1}+|\boldsymbol{u}|)$!

Implementation of the scheme in item C.

This scheme does not work either, though it does not fail as spectacularly as the scheme in item **B** — the grid scale oscillations have (comparatively) slower growth ratios. Nevertheless: grid scale oscillations are generated, which grow exponentially with time. Furthermore, the growth rate of these oscillations becomes larger as the grid is refined. See figure 7.3.

Remark 7.2 *If the scheme in item* **A** *is used under conditions that violate the C.F.L. condition in (7.10), then behavior similar to the one observed for the schemes in items* **B** *and* **C** *appears. Namely: grid scale oscillations that grow exponentially, with growth rates becoming larger as $h \to 0$.*

Note: if the C.F.L. violation is "small", then the growth rate of the oscillations is small when N is moderate. Hence, the grid scale oscillations will be hard to see, unless N is large enough, or the interval of computation (i.e.: T) is long enough.

Remark 7.3 *Not only is it possible to prove that the scheme in item* **A** *converges if the C.F.L. condition is enforced — see remark **7.1**. It is also possible to explain, theoretically, why and when grid scale oscillations will appear, and*

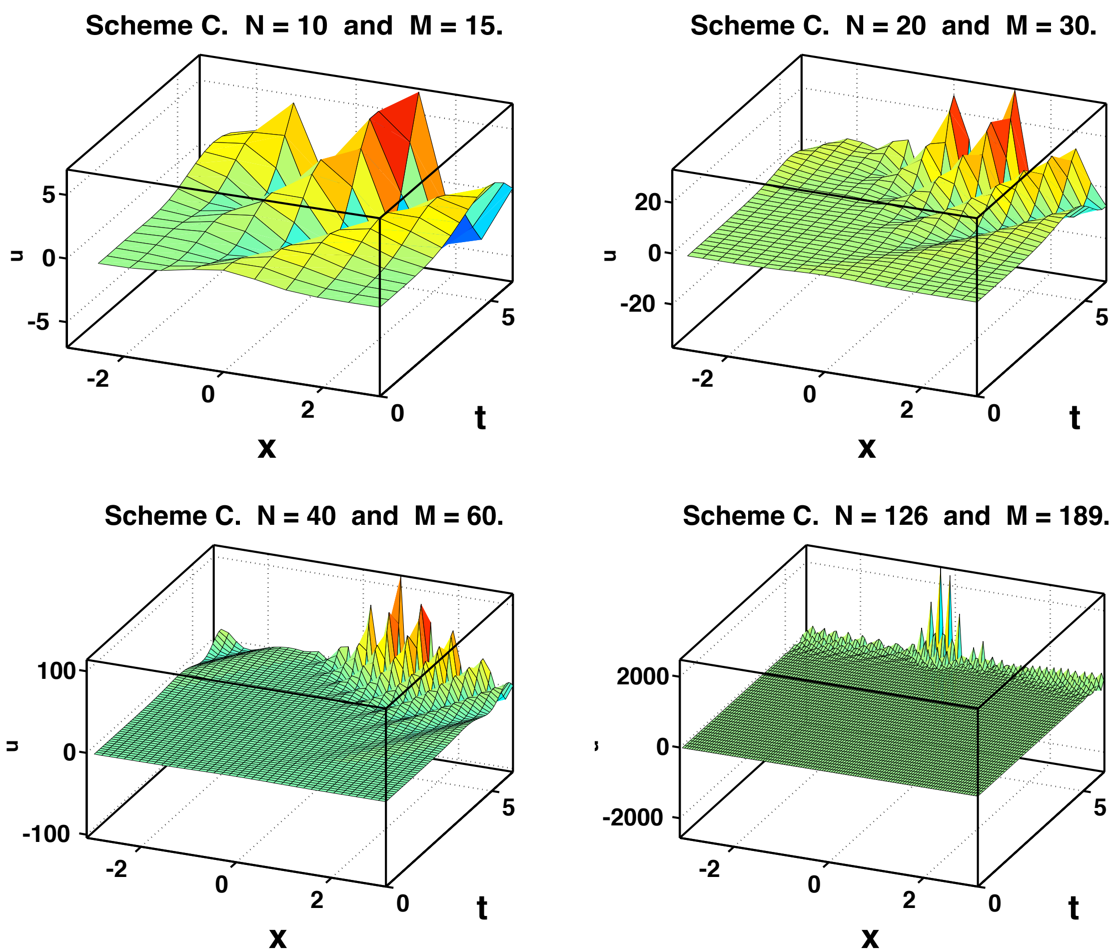


Figure 7.3: Calculations with the scheme in item **C**, to solve (7.1 – 7.2). Left to write, and top to bottom: Numerical solutions, as the grid is refined. M is substantially above the value in (7.10). Larger M's do not help.

to estimate their growth rate.

8 The flux for a conserved quantity must be a vector

8.1 Statement: The flux for a conserved quantity must be a vector

Note: Below (for simplicity) we present many arguments/questions in 2D. However they apply just as well in nD; $n = 3, 4, \ldots$

Consider some conserved quantity, with density $\rho = \rho(\vec{x}, t)$ and flux vector $\vec{q} = \vec{q}(\vec{x}, t)$ in 2D. Then, in the absence of sources or sinks, we made the argument that conservation leads to the (integral) equation

$$\frac{d}{dt} \int_{\Omega} \rho \, \mathrm{d}x_1 \mathrm{d}x_2 = - \int_{\partial\Omega} \vec{q} \cdot \hat{n} \, \mathrm{d}s, \tag{8.1}$$

for any region Ω in the domain where the conserved "stuff" resides, where $\partial\Omega$ is the boundary of Ω, s is the arc-length along $\partial\Omega$, and $\hat{n}$ is the outside unit normal to $\partial\Omega$.

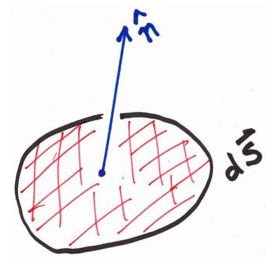
There are **two implicit assumptions** used above

1. The flux of conserved stuff is local: stuff does not vanish somewhere and re-appears elsewhere (this would not violate conservation). For most types of physical stuff this is reasonable. But one can think of situations where this is not true — e.g.: when you wire money, it disappears from your local bank account, and reappears elsewhere (with some loses due to fees, which go to other accounts).
2. The flux is given by a vector. But: $\Rightarrow$ *Why should this be so?* $\Leftarrow$ (8.2)
   The objective of this problem is to answer this question.

Given item **1**, **the flux can be characterized/defined as follows** †

$$\left.\begin{array}{l}\text{For any surface element } d\vec{S} \text{ (at a point } \vec{x}\text{, with unit normal } \hat{n}\text{)}\\ \text{the flux indicates how much stuff, per unit time and unit area,}\\ \text{crosses } d\vec{S} \text{ from one side to the other, in the direction of } \hat{n}.\end{array}\right\} \tag{8.3}$$

† This is in 3D. For a 2D, change "surface element" to "line element".



It follows that the flux should be a scalar function of position, time, and direction. That is:

$$q = q(\vec{x},\, t,\, \hat{n}), \tag{8.4}$$

where q is the amount of stuff, per unit time and unit length, crossing a curve [7] with unit normal $\hat{n}$ from one side to the other (with direction [8] given by $\hat{n}$). Then (8.1) takes the form

$$\frac{d}{dt}\int_{\Omega} \rho\,\mathrm{d}x_1\mathrm{d}x_2 = -\int_{\partial\Omega} q(\vec{x},\, t,\, \hat{n})\,\mathrm{d}s. \tag{8.5}$$

However, in this form we cannot use Gauss' theorem to transform the integral on the right over $\partial\Omega$, into one over Ω. *This is a serious problem, for this is the crucial step in reducing* (8.1) *to a pde.*

Your task: *Show that,* provided that ρ and q are "nice enough" functions (e.g.: continuous partial derivatives), *equation* (8.5) *can be used to show that q has the form*

$$q = \hat{n}\cdot\vec{q}, \tag{8.6}$$

for some vector valued function $\vec{q}(\vec{x},\, t)$.

Hints.

A. *It should be obvious that the flux going across any curve (surface in 3D) from one side to the other should be equal and of opposite sign to the flux in the opposite direction. That is, q in (8.4) satisfies*

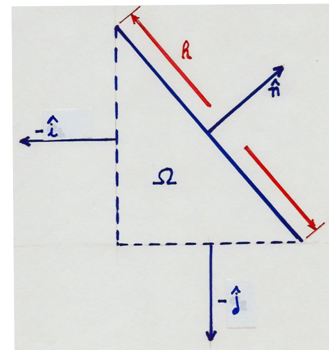
$$q(\vec{x},\, t,\, -\hat{n}) = -q(\vec{x},\, t,\, \hat{n}). \tag{8.7}$$

Violation of this would result in the conserved "stuff" accumulating (or being depleted) at a finite rate from a region with zero area (zero volume in 3D), which is not compatible with the assumption that ρ_t is continuous and equation (8.5). **Note:** there are situations where it is reasonable to make models where conserved "stuff" can have a finite density on curves or surfaces (e.g.: surfactants at the interface between two liquids, surface electric charge, etc.). Dealing with situations like this requires a slightly generalized version of the ideas behind (8.7).

B. *Given an arbitrary small curve segment* [9] *of length $h > 0$ and unit normal $\hat{n}$, realize it as the hypotenuse of a right triangle where the other sides are parallel to the coordinate axes. Then write (8.5) for the triangle, divide the result by h, and take the limit $h \downarrow 0$. Note that, if the segment of length h is parallel to one of the coordinate axis, then one of the sides of the triangle has zero length, and the triangle has zero area — but the argument still works, albeit trivially (it reduces to the argument in* **A***).*

[7] In 3D: "... and unit area, crossing a surface ..."
[8] A positive q means that the net flow is in the direction of $\hat{n}$.
[9] You can assume it is straight, since a limit $h \downarrow 0$ will occur.

8.2 Answer: The flux for a conserved quantity must be a vector



Consider a small (straight) segment of length $0 < h \ll 1$, which is not parallel to the coordinate axes. Let $\hat{n}$ be the unit normal to the segment with a positive first component $n_1 > 0$. Assume that $n_2 > 0$. Construct a right triangle Ω, with two sides parallel to the coordinate axes and hypotenuse the given segment (see figure). Then $\partial\Omega$ has outside unit normals $\hat{n}$, $-\hat{\imath}$, and $-\hat{\jmath}$ (where $\hat{\imath}$ and $\hat{\jmath}$ are the coordinate axes unit vectors), and equation (8.5) yields

$$\begin{aligned} O(h^2) &= -q(\vec{x},\, t,\, \hat{n})\, h - q(\vec{x},\, t,\, -\hat{\imath})\, h\, n_1 - q(\vec{x},\, t,\, -\hat{\jmath})\, h\, n_2 + O(h^2), \\ &= -q(\vec{x},\, t,\, \hat{n})\, h + q(\vec{x},\, t,\, \hat{\imath})\, h\, n_1 + q(\vec{x},\, t,\, \hat{\jmath})\, h\, n_2 + O(h^2), \end{aligned} \tag{8.8}$$

where $\vec{x}$ is any point in Ω and we use (8.7) to obtain the second line from the first. Now divide (8.8) by h, and take the limit $h \downarrow 0$. This yields (8.7) with

$$\vec{q} = q(\vec{x},\, t,\, \hat{\imath})\, \hat{\imath} + q(\vec{x},\, t,\, \hat{\jmath})\, \hat{\jmath}. \tag{8.9}$$

To complete the proof we need to consider the cases:

— Case $n_1 > 0$ and $n_2 < 0$. The argument is exactly analogous to the one above.

— Case $n_1 < 0$ and $n_2 \neq 0$. Follows from the result in (8.7).

— Case $n_1 = 0$ or $n_2 = 0$. Trivial, given $\vec{q}$ as in (8.9), and (8.7).

9 Traveling waves for KdV and the Airy paradox

9.1 Statement: Traveling waves for KdV and the Airy paradox

Consider the Korteweg de-Vries (KdV) equation

$$u_t + u_x + 6\, u\, u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \tag{9.1}$$

where $u = u(x,\, t)$. Before getting to the problem's questions, some needed background.

1. It seems reasonable to assume that the small amplitude solutions are well represented by the linearized equation. In particular, that we can write

$$u \approx \int_{-\infty}^{\infty} F(k)\, e^{i\,(k\,x - \omega\, t)}\, dk, \quad \text{where} \quad \omega = k - k^3, \tag{9.2}$$

and F is the Fourier Transform of the initial data $u_0(x) = u(x,\, 0)$.

Imagine now that u_0 is a localized bump [10], write F in polar form [11] $F = f\, e^{i\,\phi_0}$, and rewrite (9.2) as

$$u \approx \int_{-\infty}^{\infty} f(k)\, e^{i\,\phi + i\,k\,x}\, dk, \quad \text{where} \quad \phi = \phi_0 - \omega\, t. \tag{9.3}$$

In particular

$$u_0 = \int_{-\infty}^{\infty} f(k)\, e^{i\,\phi_0 + i\,k\,x}\, dk. \tag{9.4}$$

[10] In particular, u_0 decays very fast as $|x| \to \infty$. Say, exponentially.
[11] Here both f and ϕ_0 are real valued.

The way a Fourier Transform can represent a localized function is through destructive interference. This requires a careful balance between the amplitude f and the angle ϕ_0. However

$$\frac{d^2\phi_0}{dk^2} - \frac{d^2\phi}{dk^2} = \frac{d^2\omega}{dk^2}\,t, \quad \text{where} \quad \frac{d^2\omega}{dk^2} \neq 0. \tag{9.5}$$

Thus the required balance is destroyed by the time evolution (see note below). It follows that

The solution in (9.3), ceases to be localized in space as time increases. Oscillations appear, which spread over an ever larger region (growing linearly with time). The bump "disperses". (9.6)

Note. The reason that we need to look at the second derivative in (9.5) is that linear changes in the angle merely cause a shift in space of the transform. A wave system for which $\frac{d^2\omega}{dk^2} \neq 0$ is called *dispersive*, for the reasons explained above.

2. Yet equation (9.1) has steady traveling wave solutions (with constant speed s)

$$u = U(x - s\,t), \quad \text{where } u \text{ vanishes exponentially as } |x - s\,t| \to \infty. \tag{9.7}$$

Furthermore, these solutions satisfy $0 < U \leq a$, where the parameter $a = \max(U)$ can be any positive number — in particular, as small as you care it to be.

This is in clear contradiction with (9.6)! $\longleftarrow$ *The Airy "paradox".*

It is hard to argue with (9.7), which means that something is wrong with the argument leading to (9.6) — not too surprising since this argument is not "rigorous". Yet, there is no question that

$$\int_{-\infty}^{\infty} F(k)\, e^{i\,(k\,x - \omega\,t)}\, dk \tag{9.8}$$

exhibits the behavior in (9.6) — this is rigorous. Thus the problem must be with $\approx$ in (9.2). *The question is what, and is it something with a clean intuitive interpretation,* or is this a problem with a subtle and very technical, but not intuitive, mathematical reason behind it?

PROBLEM TASKS:

A. Write the solutions in (9.7), parameterized by a. This involves elementary functions only.

B. Study the way U scales with a, and provide an intuitive explanation for the Airy paradox.
Hint. What happens with the dispersive effects as a vanishes?

9.2 Answer: Traveling waves for KdV and the Airy paradox

Upon substitution of (9.7) into (9.1), the equation becomes an third order ode for U. Using the fact that U vanishes very fast at infinity, this ode can be integrated twice, to obtain

$$(U')^2 + (1 - s)\,U^2 + 2\,U^3 = 0, \tag{9.9}$$

where the prime indicates a derivative with respect to $x - s\,t$. This equation is separable, and can be solved using elementary functions. After a bit of algebra, this yields the solutions

$$u = a\,\mathrm{sech}^2\left\{\sqrt{a}\,[x - (1 + 4\,a)\;t]\right\}, \tag{9.10}$$

where $a > 0$ is an arbitrary constant. As advertised, these solutions decay exponentially at infinity, and remain "a bump" for all times — no matter how small a is taken.

Next we use the solution above to motivate an intuitive explanation for the paradox.

a. Notice that, as the amplitude of (9.10) decreases, the "wavelength" $O(1/\sqrt{a})$ correspondingly increases. Since the term responsible for the dispersion in (9.1) is u_{xxx}, we see that for (9.10) dispersion effects vanish as the nonlinearity decreases. In fact, a quick calculation shows that

$$\text{"size" of the nonlinear effects} \quad = \text{"size" of} \quad u\,u_x \quad = \quad a^{\frac{5}{2}}. \tag{9.11}$$

$$\text{"size" of the dispersive effects} \quad = \text{"size" of} \quad u_{xxx} \quad = \quad a^{\frac{5}{2}}. \tag{9.12}$$

It is thus

not valid to neglect the nonlinear effects when going from (9.1) to (9.2), while keeping the dispersive ones — if the wavelength is sufficiently large. (9.13)

The moral here is that, to linearize an equation (particularly one with dispersion) we not only need to assume that the amplitudes are small, but we need to watch the wavelengths as the amplitude vanishes — unforeseen errors can occur if we do not.

b. A more "mathematical" way to see this is to observe that (9.1) has the following symmetry

$$\text{If } \; u = u(x,\, t) \; \text{ solves (9.1), then so does } \; \tilde{u} = m^2\, u\left(m\,(x-t) + m^3\,t, m^3\,t\right), \tag{9.14}$$

where m is any constant.[12] Thus it is possible to generate solutions with arbitrarily small amplitudes such that the nonlinear term in (9.1) cannot be neglected.

Finally, we point out that the Airy paradox did not arise in the context of the KdV equation, but for the full water waves equations. These equations do not have a simple invariance such as (9.14), and quantifying the "size" of the nonlinear and the dispersive effects is not as easy. Hence, it took quite a while for the paradox to be resolved. In fact, the key ingredient in doing so was the derivation of the KdV equation as applying, for water waves, in a special limit involving small amplitude and long waves [after nondimensionalizing using the fluid depth]. This limit is precisely the one in which the amplitude scales as the square of the inverse of the wavelength — thus, it is not surprising that the KdV equation should be invariant under this scaling, as in (9.14).

THE END.

[12] The solutions in (9.10) are generated in this fashion, with $a = m^2$, from the solution corresponding to $a = 1$.