Answers to Problem Set # 01, 18.306
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Contents

1 Fundamental Diagram of Traffic Flow #03

1.1 Statement: Fundamental Diagram of Traffic Flow #03

Many state laws state that: for each 10 mph (16 kph) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 ft (5 m)). There is another law that gives a maximum speed limit (assume that this is 50 mph (80 kph)). **Find the flow of cars as a function of density, $q = q(\rho)$**, that results from these two laws.

The state laws on following distances stated in the prior paragraph were developed in order to prescribe a spacing between cars such that rear-end collisions could be avoided, as follows:

a. Assume that a car stops instantaneously. *How far would the car following it travel* if moving at u mph and
a1. The driver's reaction time is τ, and
a2. After a delay τ, the car slows down at a constant maximum deceleration α.

b. The calculation in part **a** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate α, but the driver in the following car still takes a time τ to react. *How far back does a car have to be, traveling at u mph, in order to prevent a rear-end collision?*

c. Show that the law described in the first paragraph of this problem corresponds to part **b**, if the human reaction time is about 1 sec. and the length of a car is about 16 ft (5 m).

Note: *What part* **c** *is asking you to do is to justify/derive the state law prescription, using the calculations in part* **b** *to arrive at the minimum car-to-car separation needed to avoid a collision when the cars are forced to brake.*

1.2 Answer: Fundamental Diagram of Traffic Flow #03

Assume that the drivers follow the state law prescription exactly. Then [1]

$$d = \frac{L\,u}{V}, \tag{1.1}$$

where d is the distance to the next car, L is the car length, u is the car velocity and V is the law "trigger" velocity, as in:

State Law: *Maintain a distance of one car length for each V increase in velocity.*

Typical numbers are $V = 10$ mph $= 16$ kph and $L = 16$ ft $= 5$ m.

Since we then end up with one car for every $d + L$ distance, equation (1.1) above leads to the velocity–density relationship

$$\frac{1}{\rho} = L + d = L\left(1 + \frac{u}{V}\right) \quad \text{or} \quad u = \left(\frac{1}{\rho\,L} - 1\right) V. \tag{1.2}$$

Thus the car flux is given by

$$q = u\,\rho = \left(\frac{1}{L} - \rho\right) V \implies c = \frac{dq}{d\rho} = -V. \tag{1.3}$$

This gives a constant wave speed c, and a car velocity that goes to infinity as the density vanishes. This because we have not yet enforced the **speed limit,** which yields

$$u = \min\left\{u_m,\ \left(\frac{1}{\rho\,L} - 1\right) V\right\} \quad \text{and} \quad q = \min\left\{\rho\,u_m,\ \left(\frac{1}{L} - \rho\right) V\right\}, \tag{1.4}$$

where $u_m =$ speed limit. The critical density below which $u = u_m$, is

$$\rho_c = \frac{V}{L\,(V + u_M)}. \tag{1.5}$$

Then (1.2 – 1.3) applies for $\rho \geq \rho_c$. Then, if $u < u_m$ we can recover $\rho > \rho_c$ from (1.2 – 1.3). On the other hand, $u = u_m$ for any $\rho \leq \rho_c$.

[1] To understand this note that d/L is the number of car lengths of separation, while u/V is the number of "trigger" velocities.

Next we motivate the state laws by a simple calculation involving two facts: (i) Drivers have a finite reaction time, $\tau > 0$. (ii) Cars do not change velocity instantaneously, but do so with a finite deceleration $\alpha > 0$. To simplify matters, here we assume that τ is the same for all the drivers, and that α is constant (and the same for all cars). Imagine now two cars, one behind the other, traveling at the same (constant) speed u. At some point, the car ahead (car #1) starts braking. It then travels a distance

$$D_1 = \frac{1}{2}\frac{u^2}{\alpha} \tag{1.6}$$

from the moment the brakes are applied to the moment it stops. On the other hand, the distance traveled by the car behind (car #2) from the moment the driver sees that he must stop [2] till the car actually stops, includes the driver's reaction time. That is

$$D_2 = u\,\tau + \frac{1}{2}\frac{u^2}{\alpha}. \tag{1.7}$$

It then follows that, *in order to avoid a rear end collision, the distance between two cars traveling at speed $\boldsymbol{u}$ must be at least $u\,\tau$*. This yields the following formula for the "trigger" velocity

$$V = \frac{L}{\tau}. \tag{1.8}$$

In particular, $L = 16$ ft and $\tau = 1$ s yields $V = 16$ ft/s $= 10.9$ mph — note that 1 mile $= 5280$ ft and 1 hr $= 3600$ s. This trigger velocity is pretty close to the one used in many state laws.

Note: The calculation above is a bit sloppy, for it only checks that car #2 is still behind car #1 once they stop. What one should check is that car #2 stays behind car #1 at *all* times. However car #2 is always moving at a speed equal to or greater than that of car #1. This because it starts slowing down, at the same rate and starting from the same speed, later. Thus the distance between the two cars is a non-increasing function of time. It follows that it is enough to check that it is positive once they stop, to know that it was always positive.

2 Ill posed Laplace equation problem #01

2.1 Statement: Ill posed Laplace equation problem #01

Consider the following problem involving Laplace's equation

$$u_{xx} + u_{yy} = 0, \tag{2.1}$$

on the strip $0 < x < 2\,\pi$ and $-\infty < y < \infty$:

Given $u(0,\,y) = f(y)$ and $u_x(0,\,y) = g(y)$, determine $u(2\pi,\,y) = h(y)$ — where f and g are smooth periodic functions.

Show that this is an ill-posed problem.

Hint: Consider what happens with high frequency perturbations.

2.2 Answer: Ill posed Laplace equation problem #01

Let $u = u_*(x,\,y)$ be a solution to the given problem, for some $f = f_*(y)$ and $g = g_*(y)$, where $h = h_*(y) = u_*(2\pi,\,y)$. Without loss of generality, assume that the period in y is $2\,\pi$.

[2] Say, the brake lights for the car ahead turn on.

Consider a sequence (one for each $n = 1,\, 2,\, 3\dots$) of perturbed problems with $\boldsymbol{f = f_* + \epsilon_n \sin(ny) = f_* + (\Delta f)_n}$ and $\boldsymbol{g = g_* + n\,\epsilon_n \sin(ny) = g_* + (\Delta g)_n}$, where ϵ_n is a scalar sequence such that $\boldsymbol{\epsilon_n \to 0}$ as $\boldsymbol{n \to 0}$ (thus the perturbations vanish as $n \to 0$).

Then
$$u = u_* + \epsilon_n\, e^{n\,x}\, \sin(n\,y), \tag{2.2}$$
which yields
$$h = h_* + \epsilon_n\, e^{2\,n\,\pi}\, \sin(n\,y) = h_* + (\Delta h)_n \tag{2.3}$$

Now we notice that we can select the ϵ_n so that: *as $n \to \infty$, the size of the perturbation $(\Delta h)_n$ to the answer h* **does not** *vanish.*† Hence **the answer does not depend continuously on the data,** so the problem is ill-posed.

† For example, take $\boldsymbol{\epsilon_n = e^{-n\,\pi}}$, in which case $(\Delta h)_n$ not only does not vanish as $n \to \infty$, but goes to infinity!

Remark 2.1 *In addition, notice that there are also problems with the existence of solutions. For example, if f is given by the Fourier series $f = \sum_n f_n \sin(n\,y)$, and $g = 0$, the solution u can be written in terms of the Fourier series*
$$u = \sum_n f_n\, \cosh(n\,x)\, \sin(n\,y). \tag{2.4}$$
However, this series will not converge for $x = 2\,\pi$ (in fact, any $x > 0$) unless the Fourier coefficients f_n decay exponentially fast as $n \to \infty$ — implying that there may not even a solution unless f is analytic on a strip of width $4\,\pi$ around the real y axis.

Remark 2.2 (Mathematical subtle point 1). *Notice that we have not defined what we mean by the "size" of a function.*‡ *Since there are many (non-equivalent) ways of assigning a size to a function, we should worry that perhaps there is some way that will destroy the argument above. However, notice that $(\Delta g)_n = n\,(\Delta f)_n$ and $(\Delta h)_n = e^{2\,n\,\pi}\,(\Delta f)_n$. Hence, as long as* **(A-C)** *below apply,*[3] *the argument works — e.g.: select $\epsilon_n = e^{-n\,\pi}/\|\sin(n\,y)\|$.*

‡ "Size" is needed to give meaning to $(\Delta f)_n \to 0$. If $\|f\|$ is the size of f, then $(\Delta f)_n \to 0$ means $\|(\Delta f)_n\| \to 0$.
$\|f\|$ is called a "norm", and we require:

(A) $\|f\| \geq 0$; **(B)** $\|f\| = 0 \iff f = 0$; **(C)** For α scalar, $\|\alpha\, f\| = |\alpha|\, \|f\|$; and **(D)** $\|f + g\| \leq \|f\| + \|g\|$.

A few examples are: $\|f\|_\infty = \sup |f(y)|$, $\|f\|_1 = \int |f(y)|\mathrm{d}y$, and $\|f\|_2 = \sqrt{\int |f(y)|^2 \mathrm{d}y}$. Note that each norm applies to a set of admissible functions only. Some norms include derivatives and requiere functions with derivatives.

Remark 2.3 (Mathematical subtle point 2, but with practical consequences). *On the other hand, we can destroy the argument by restricting the class of allowed data functions f and g. For example, if we only allow f and g to have a finite number of Fourier coefficients*
$$f = \sum_{n=-N}^{n=N} f_n\, e^{i\,n\,y} \quad \text{and} \quad g = \sum_{n=-N}^{n=N} g_n\, e^{i\,n\,y}, \tag{2.5}$$
with no frequency above some fixed threshold N, then the problem becomes well posed.[4]
The above, of course, makes the problem into a finite dimensional one, and no longer a p.d.e. problem. Trivial as this example may be, it shows that ill-posed problems can be turned into well-posed ones by suitable restrictions/changes. An interesting example of this occurs with CAT scanning, whereby the objective is to reconstruct the density $\rho = \rho(x,\, y)$ of a cross-section of a body from the amount of damping produced on x-rays traveling across the cross-section in many different directions. As it turns out, if one insists on recovering the point values of the density, the problem is ill-posed. However, if one only attempts to recover density local averages (i.e.: a suitably "smeared" density), the problem becomes quite reasonable. The analog of this for "our" problem here would be to apply a "filter" to the data f and g, so any high frequencies they have are removed, and then only request to recover a suitably filtered h. This, pretty much, would transform the problem into the (well posed) version above in (2.5).

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[3] Also: $\|sin(ny)\|$ does not grow too fast with n — norms that heavily penalize high frequencies are possible!
[4] Effectively, in terms of the notation in remark **2.2**, this amounts to setting "$\|e^{i\,n\,y}\|_n = \infty$" for $|n| > N$.

3 Laplace equation problem #01

3.1 Statement: Laplace equation problem #01

Consider a thin, homogeneous, heat conducting sheet, insulated on the top and the bottom. Then, if $T = T(x,\, y,\, t)$ is the temperature in the sheet, the conservation of heat (and Fick's law) leads to the heat equation — which in non-dimensional units has the form

$$T_t = \Delta\, T = T_{xx} + T_{yy}. \tag{3.1}$$

Let Ω be the region of space occupied by the sheet, and assume that along the boundary $\partial\Omega$ of this region the heat flux is known and given by some function, say: $F = F(s)$ per unit length (where s is the arc-length along $\partial\Omega$).

The problem to be solved is then (3.1) inside Ω, with the boundary conditions on $\partial\Omega$

$$\partial_n T = \hat{n}\cdot\nabla T = F(s), \quad \text{where } \hat{n} = \text{unit outside normal to } \partial\Omega. \tag{3.2}$$

In particular, for *steady state*, we have Laplace's equation in Ω: $$0 = \Delta\, T = T_{xx} + T_{yy}, \tag{3.3}$$
with the *Neumann* boundary condition in (3.2).

1. Show that there is an integral condition that F must satisfy if the problem (3.2–3.3) has a solution. **Hint:** *Gauss theorem.*
2. Give a physical interpretation to the condition in **1.** Why do you need it, and what happens when it is not satisfied?
3. The solution to (3.2–3.3), if there is one, is determined only up to an arbitrary additive constant. How would you determine this constant, and what is it related to — i.e.: knowledge of what physical quantity gives it to you?

3.2 Answer: Laplace equation problem #01

1. We have
$$0 = \int_\Omega \Delta T\, dA = \int_{\partial\Omega} \partial_n T\, ds = \int_{\partial\Omega} F(s)\, ds, \tag{3.4}$$
where dA is the element of area in Ω, ds is the arc-length on $\partial\Omega$, and we have used Gauss theorem — note that $\Delta = \text{div}\,\nabla$. Hence, for (3.2–3.3) to have a solution F must satisfy
$$0 = \int_{\partial\Omega} F(s)\, ds. \tag{3.5}$$
2. The integral in equation (3.5) is the *net heat flux (positive when out) across $\partial\Omega$.* Clearly, no steady state can be achieved if the net heat flux does not vanish, as otherwise the amount of heat contained within Ω will either increase or decrease steadily.
3. In order to find the arbitrary constant up to which the solution to (3.2–3.3) is determined, we need to know (for example) the average temperature (equivalent to knowing the total amount of heat [5] Q inside Ω). If we think of the solution to (3.2–3.3) as the steady state limit (as $t \to \infty$) of the solution to (3.1–3.2) — plus initial conditions, this knowledge comes from the initial conditions. It is the only piece of information from the initial conditions that the heat distribution "remembers" after a long time. The heat equation is "irreversible" and it destroys information with its evolution [just the same as shocks do: characteristics end at shocks, and the information they carry is lost].

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[5] If the total heat flux across the boundary vanishes — needed to have a steady state, Q is constant.

4 Nonlinear solvable ODEs

4.1 Statement: A nonlinear solvable ode #01

Generally nonlinear ode, even simple scalar and first order ones, do not admit explicit exact solutions. There are, however, some exceptions. For example, consider the ode

$$0 = 1 + \frac{1}{y'} + \left(x - \frac{y}{y'}\right)^2, \tag{4.1}$$

where $y' = \dfrac{dy}{dx}$. This ode is nonlinear with variable coefficients, and not separable. Yet it can be solved exactly, even though the standard methods do not apply. Your **task** here is to do this.

Hint. Introduce the variables,

$$u = \frac{1}{y'} \quad \text{and} \quad v = x - \frac{y}{y'} = x - y\,u, \tag{4.2}$$

and investigate what equation (4.1) implies for them.

4.2 Answer: A nonlinear solvable ode #01

From (4.1) and (4.2) it follows that

$$0 = 1 + u + v^2. \tag{4.3}$$

Furthermore

$$\frac{dv}{dx} = -y\,\frac{du}{dx} = 2\,y\,v\,\frac{dv}{dx}, \tag{4.4}$$

where the first equality follows from the definition of u and v, and the second upon use of (4.3). It follows that, either $\boldsymbol{dv/dx = 0}$ or $\boldsymbol{2\,y\,v = 1}$. We analyze these two cases below.

The case $dv/dx = 0$.

In this case $v = x_0$ is a constant. Using this fact in the definition of v in (4.2) yields

$$\frac{y'}{y} = \frac{1}{x - x_0} \quad \Longrightarrow \quad y = c\,(x - x_0), \tag{4.5}$$

where c is some constant. Substituting this back into equation (4.1) yields

$$0 = 1 + \frac{1}{c} + x_0^2. \tag{4.6}$$

Hence we arrive at the **family of solutions to (4.1)** given by

$$\boldsymbol{y = -\frac{x - x_0}{1 + x_0^2}}, \tag{4.7}$$

where x_0 is an arbitrary constant.

The case $2\,y\,v = 1$.

Then $v = 1/(2\,y)$ and, using (4.3), $u = -1 - 1/(4\,y^2)$. But, from (4.2), $v = x - y\,u$. Hence

$$\frac{1}{2\,y} = x + y\left(1 + \frac{1}{4\,y^2}\right) \quad \Longleftrightarrow \quad 0 = x + y - \frac{1}{4\,y}. \tag{4.8}$$

We conclude that

$$\boldsymbol{y = \frac{-x \pm \sqrt{1 + x^2}}{2}}. \tag{4.9}$$

To verify that **these are indeed solutions,** note that the equality on the right in (4.8) yields, upon differentiation

$$0 = 1 + \left(1 + \frac{1}{4\,y^2}\right) y' \quad \Longrightarrow \quad u = -1 - \frac{1}{4\,y^2}, \tag{4.10}$$

using the definition of u in (4.2). But then we can write the equality on the left in (4.8) in the form

$$\frac{1}{2\,y} = 1 - y\,u = v, \tag{4.11}$$

where we have used the definition of v in (4.2). Finally, substituting (4.11) into (4.10) yields (4.3), which is precisely what (4.1) looks like in terms of u and v.

5 Small vibrations of a 2D string under tension (sv2ds)

5.1 Introduction: Small vibrations of a 2D string under tension

Here you are asked to derive *equations for the transversal vibrations of a thin elastic string under tension*, under several scenarios. The following hypotheses are common to all of them (5.1)

1. The string is homogeneous, with *mass density* (mass per unit length) $\rho =$ constant.
2. The motion is restricted to the x-y plane. At equilibrium the string is described by $y = 0$ and $0 \leq x \leq L =$ string length. The *tension* $T > 0$ is then *constant*. See remark **5.1**, #1–3.
3. The string has no bending strength.
4. The amplitude of the vibrations is very small compared with their wavelength, and any longitudinal motion can be neglected. Thus:
   — The string can be described in terms of a *deformation function* $u = u(x,\, t)$,
   such that the equation for the *curve describing the string* is[6] $y = u(x,\, t)$.
   — The tension remains constant throughout — see remark **5.1**, #4.

Remark 5.1 *Some details:*

#1 *We idealize the string as a curve. This is justified as long as $\lambda \gg d$, where $\lambda =$ scale over which motion occurs, and $d =$ string diameter. This condition justifies item* **3** *as well.*

#2 *The tension is generated by the elastic forces (assume that the string is stretched). At a point along the string, the tension is the force with which each side (to the right or left of the point) pulls on the other side,[7] and it is directed along the direction tangent to the string (there are no normal forces, see item* **3***).*

#3 *At equilibrium the tension must be constant. For imagine that the tension is different at two points $a < b$ along the string. Then the segment $a \leq x \leq b$ would receive a net horizontal force (the difference in the tension values), and thus could not be at equilibrium.*

#4 *The hypotheses imply that string length changes can be neglected. Thus the stretching generating the tension does not change significantly, and the tension remains equal to the equilibrium tension T.*

5.2 Statement: sv2ds04 (string on elastic bed)

Here we consider the simple situation, where (on the string) there is no other force beside the tension T, nor any dissipation. However, the string is attached to an elastic bed, which produces an elastic restoring force (per unit length) pulling the string towards its equilibrium

[6] In this approximation the x-coordinate of any mass point on the string does not change in time.
[7] If you were to cut the string, this would be the force needed to keep the lips of the cut from separating.

position. Thus this force per unit length has the simple form
$$-k_b\, u, \tag{5.2}$$
where $k_b > 0$ is the spring bed constant. Under these conditions, plus those in § 5.1:

Task #1. Use the conservation of the transversal momentum to derive an equation for u. Thus:

i) Obtain a formula for the transversal momentum density along the string (momentum per unit length).
ii) Obtain a formula for the transversal momentum flux along the string (momentum per unit time).
iii) Use the differential form of the conservation of transversal momentum to write a pde for u.
Note that there is a momentum source as well.

Hint. Remember that internal string forces[8] *are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces.* Be careful with the sign here! *Remember also that u_x is very small.*

Task #2. Use the conservation of energy to derive another equation for u. Furthermore, show that the solutions to the equation derived in task #1 satisfy the conservation of energy equation.

Hint. The energy density is the sum of the kinetic energy per unit length (e_k), the elastic energy per unit length stored in the bed (e_b), and the elastic energy per unit length stored in the string deformation (e_d). Expressions for the first two are fairly easy to obtain. Let us now consider how to get an expression for e_d. In fact, you only need to compute the difference between the elastic energy, and some constant energy — specifically: the elastic energy of the string at equilibrium. But the tension is constant (see § 5.1) over the whole stretching process taking the string from equilibrium $y = 0$ to $y = u(x, t)$. Thus $e_d\, dx$ is the product of T times the change in length of the segment dx. All you need now is the formula for the change in length of any interval dx (actually, just the leading order approximation to this, in the u_x small limit).

This yields $e_d = T\, \left(\sqrt{1+u_x^2} - 1\right) = \frac{1}{2}\, T\, u_x^2$, since the arc-length is $ds = \sqrt{1+u_x^2}\, dx$, and u_x is small.

Task #3. Assuming that the string is tied at both ends $u(0,\, t) = u(L,\, t) = 0$, write an equation for the evolution of the total energy. That is: $\frac{d\mathcal{E}}{dt}$ =??

5.3 Answer: sv2ds04

We have

i) The *transversal momentum per unit length* along the string is given by $\rho_{\text{mom}} = \rho\, u_t$.

ii) *Momentum flux along the string* happens because of transversal forces.
That is: the force in the y-direction that, at every point, one side of the string applies on the other side. This is just the y-component of the tension, which is given by $T\,\sin\theta$, where θ is the angle that the string tangent makes with the x-axis. However, since u_x is small, $\theta \approx u_x$. Thus the transversal momentum flux is given by $q_{\text{mom}} = -T\, u_x$.
Note about the sign here: when $u_x < 0$, the left side of the string pulls the right side up, generating a positive momentum flux. Vice-versa, when $u_x > 0$, momentum flows from right to left (negative). Thus the sign is as above.
Note also that the x-component of the tension (horizontal force along the string) is $T\,\cos\theta = T$, since u_x is small. Thus it is constant, consistent with the approximation of no motion in the x direction.

In addition, there is a *momentum source* (the restoring force from the elastic bead), given by (5.2).

iii) **Equation for the conservation of the transversal momentum**
It must be $(\rho_{\text{mom}})_t + (q_{\text{mom}})_x =$ sources/sinks. Thus
$$\rho\, u_{tt} - T\, u_{xx} = -k_b\, u. \tag{5.3}$$
Alternatively
$$u_{tt} - c^2\, u_{xx} = -\kappa\, u, \tag{5.4}$$
where $c^2 = T/\rho =$ wave speed and $\kappa = k_b/\rho$.
Here $\sqrt{\kappa}$ is the bed frequency (solutions with no space dependence have this frequency).

[8] Internal forces = forces by one part of the string on another.

iv) **Equation for the conservation of energy**

Using the hint for task #2 we see that $\boldsymbol{e_d = T\,\left(\sqrt{1+u_x^2}-1\right) = \frac{1}{2}\,T\,u_x^2}$,
since the arc-length is $ds = \sqrt{1+u_x^2}\,dx$, and u_x is small.
The kinetic energy per unit length is $\boldsymbol{e_k = \frac{1}{2}\,\rho\,u_t^2}$.
The bed elastic energy per unit length is $\boldsymbol{e_b = \frac{1}{2}\,k_b^2\,u}$.
The **energy flux q_e** is given by the work per unit time done by the transversal force (see (ii) above). Thus $\boldsymbol{q_e = -T\,u_x\,u_t}$.
Note that there is neither transversal momentum flux, nor energy flux, due to the bed forces. Fluxes are caused by internal forces (forces by one part of the string on another). Putting this all together, we obtain the equation for the conservation of energy

$$\boldsymbol{\left(\frac{1}{2}\,\rho\,u_t^2 + \frac{1}{2}\,T\,u_x^2 + \frac{1}{2}\,k_b^2\,u^2\right)_t - (T\,u_x\,u_t)_x = 0.} \tag{5.5}$$

It is easy to see that (5.3) guarantees this.

Why is energy conservation "automatic" for this model? The reason is that, in this model, there is no mechanism for energy exchanges between mechanical and internal *in the string*, thus there is no internal energy contribution to the motion.

v) **Time evolution of the total energy for a string tied at both ends, with $u = 0$ there.**

The total energy is $\boldsymbol{\mathcal{E} = \int_0^L \left(\frac{1}{2}\,\rho\,u_t^2 + \frac{1}{2}\,T\,u_x^2 + \frac{1}{2}\,k_b^2\,u^2\right)\,dx}$, and

$$\boldsymbol{\frac{d\mathcal{E}}{dt} = 0}, \tag{5.6}$$

as follows from (5.5). Note that this formula remains valid if $u = 0$ is replaced at either (or both) ends by $u_x = 0$.

Remark 5.2 *In item (v) above, imagine that the left end of the string is free to slide along a rod, but that there is friction opposing the sliding. Then the boundary condition $\boldsymbol{b_f\,u_t(0,\,t) = T\,u_x(0,\,t)}$ applies, where $\boldsymbol{b_f > 0}$ is the friction coefficient. In this case*

$$\boldsymbol{\frac{d\mathcal{E}}{dt} = -b_f\,u_t^2(0,\,t) \leq 0}, \tag{5.7}$$

which encodes the extra dissipation produced by friction along the rod. The extra term arises from the energy flux at the origin, given by $-T\,u_x\,u_t = -b_f\,u_t^2$. A similar equation arises if the right end of the string is allowed to slide over a rod.

5.3.1 Conservation of other quantities

In the derivation above we obtained an equation for the string using, solely, the conservation of the transversal momentum. We already saw that then energy is conserved as well. There are other quantities that should be conserved as well. Let us check that this is so.

Conservation of mass. This is guaranteed by the parameterization we use. That is, each point $(x,\,u)$ follows a specific bit of the spring, with mass $\rho\,dx$. Mass is automatically conserved. This is typical of solid mechanics problems, where the parameterization tracks points in the object. In fluids individual particles are not tracked; instead the flow velocity at each point in space is prescribed. Thus mass conservation is not automatic, and must be enforced. On the other hand, in fluids the equations involve only first time derivatives of the densities — while in solids second order equations occur. This is because the velocities are variables in fluid problems, while in solids they result from the time derivatives of the displacements.

Conservation of longitudinal momentum. Since there is no motion in the x-direction, and the horizontal component of the tension forces is constant (see the end of item (ii) of the answer), the longitudinal momentum is conserved as well (trivially so). In fact, *the statement that the derivation above does not use the conservation of the longitudinal momentum is not quite correct!* This is used, effectively, when concluding that the tension on the string is constant.

5.3.2 Simple uniqueness proof for the initial value problem

Let u_1 and u_2 be two solutions of (5.3) with the same initial values, and the same Dirichlet/Neumann boundary conditions: **(L)** $u_1(0,\,t) = u_2(0,\,t)$ or $(u_1)_x(0,\,t) = (u_2)_x(0,\,t)$; and
(R) $u_1(L,\,t) = u_2(L,\,t)$ or $(u_1)_x(L,\,t) = (u_2)_x(L,\,t)$.
Assume also that u_1 and u_2 are twice continuously differentiable (this for simplicity). Then $\boldsymbol{u_1 = u_2}$.

The proof is very simple. Because the problem is linear, $u = u_1 - u_2$ satisfies the homogeneous problem, thus (5.6) applies. But then, since $\mathcal{E}(0) = 0$, it follows that $\mathcal{E} \equiv 0$. Hence $u \equiv 0$. ♣

Note that, because of remark **5.2**, uniqueness also applies if boundary conditions of the form $b_f\, u_t(0,\,t) - T\, u_x(0,\,t) = \sigma_L(t)$ and/or $b_f\, u_t(L,\,t) + T\, u_x(L,\,t) = \sigma_R(t)$ are used.

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6 Well and/or ill posed ODE #01

6.1 Statement: Well and/or ill posed ODE #01

Consider the two ode problems [9]

$$\frac{dx}{dt} = \sigma\,\mathrm{sign}(x)\,\sqrt{|x|} \quad \text{and} \quad x(0) = 0, \tag{6.1}$$

where either $\sigma = 1$, or $\sigma = -1$. One of these problems is well posed, and the other is ill posed. **Show this.**
Hint #1. To get some intuition, draw the the slope field in the x versus t plane.
Hint #2. Consider the solutions obtained by separating variables. Are there any other solutions?
Hint #3. Note that, if:
(a) $x = x_1(t)$ solves the ode for $0 \le t \le t_0$,
(b) $x = x_2(t)$ solves the ode for $t_0 \le t$, and
(c) $x_1(t_0) = x_2(t_0)$,
Then $x(t) = x_1(t)$ for $0 \le t \le t_0$, and $x(t) = x_2(t)$ for $t_0 \le t$, solves the ode for all $t \ge 0$.

6.2 Answer: Well and/or ill posed ODE #01

1. **Case $\boldsymbol{\sigma = 1}$.** Then separation of variables (when $x \ne 0$) shows that $x = \pm\frac{1}{4}\,(t - t_0)^2$ is a solution, as long as $t \ge t_0$ — where t_0 is any constant. It follows that $x = x(t)$ (as defined below) is a solution to (6.1), with initial value $x(0) = 0$, for any t_0 and ν. Hence **the problem is ill-posed.**

   Define $x = x(t)$ by:

   $$x(t) = 0 \quad \text{for } \ 0 \le t \le t_0 \quad \text{and} \quad x(t) = \nu\,\frac{1}{4}\,(t - t_0)^2 \quad \text{for } \ t \ge t_0, \tag{6.2}$$

   where $t_0 \ge 0$ is an arbitrary constant, and either $\nu = 1$ or $\nu = -1$.

2. **Case $\boldsymbol{\sigma = -1}$.** Then the solution is $x \equiv 0$, and there is no other. Hence **this problem is well-posed.**

   To show that there cannot be any other solution but $x \equiv 0$, we note that $\dot{x} < 0$ if $x > 0$, and $\dot{x} > 0$ if $x < 0$. Thus the solutions are decreasing for $x > 0$, and increasing for $x < 0$. Hence, it cannot be $x(t) > 0$ for any $t > 0$, for this would imply $x(0) > 0$. Similarly, it cannot be $x(t) < 0$ for any $t > 0$, for this would imply $x(0) < 0$.

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[9] Note that the ode right hand side is a continuous function of x, in spite of the sign function present there.

7 Well and/or ill posed ODE #02

7.1 Statement: Well and/or ill posed ODE #02

Consider a **cylindrical water bucket,** loosing its contents through a small, but not too small,[10] hole in the bottom. Assume also that no water is being added at the top.
In this problem you are asked to:

1. **Derive an equation** for the height of the water level in the bucket.
2. **Give a physical interpretation** to the results from the exercise in §6.1. In particular, show that the lack of uniqueness is natural and obvious, not pathological.

Let $h = h(t) =$ water height in the bucket, $a =$ small hole cross-sectional area, $A =$ bucket cross-sectional area (constant), $v = v(t) =$ water velocity through the hole, from inside to outside $[v \geq 0]$, $\rho =$ water density, and $g =$ acceleration of gravity. Now **derive an ode for** $\boldsymbol{h}$, as follows:

a. Use the conservation of the water volume to write an equation that relates $\dot{h}$ to v.

b. Use the conservation of energy to write an equation that relates v to h. Proceed as follows:

1. Write a formula for the gravitational potential energy, V, of the water in the bucket.
2. Write a formula for the kinetic energy flux, K_f, produced by the stream of water through the hole at the bottom.
3. Neglect dissipation, so that all the potential energy loss (as the water level falls) is converted into kinetic energy.

This will yield the desired equation.

c. Alternatively, obtain an equation relating v to h as follows **(do BOTH b and c)**

1. Assume that the pressure in the bucket is hydrostatic.
2. Assume Bernoulli's law, so that $p = p_a + \frac{1}{2}\rho v^2$, where p is the pressure at the bottom of the bucket and p_a is the pressure outside the bucket.[11]

These assumptions, as well as those in **b**, are crucially dependent on the hole being small.

d. Combine **a** and **b** to get the desired ode.

Compare the ode you just obtained with (6.1), and **give a physical interpretation to the two cases considered in** § 6.1 — i.e. $\sigma = \pm 1$.

7.2 Answer: Well and/or ill posed ODE #02

Below we show that the water height in the bucket, $h = h(t)$ satisfies the equation

$$\dot{h} = -C\sqrt{h}, \quad \text{where} \quad C = \frac{a}{A}\sqrt{2\,g} > 0 \quad \text{and} \quad h \geq 0. \tag{7.1}$$

The problem in (6.1), with $\sigma = 1$, corresponds to finding the bucket empty, and asking (for example) at what prior time was the bucket full. With the information given, this question cannot be answered. The bucket could have finished emptying a minute ago, ten minutes ago, or one hour ago. This is reflected by the infinitely many possible solutions in (6.2) — with $\nu = 1$, since we know it must be $h \geq 0$. *The lack of a unique solution is here natural and obvious, not pathological.*
On the other hand, (6.1) (with $\sigma = -1$) corresponds to finding the bucket empty, and asking what the water level will be in the future. With the assumptions made, this has the obvious answer: the bucket will stay empty. The spilled water will not, spontaneously, flow back into the bucket.

We now proceed to derive the equation for h.

[10] If the hole is too small, viscosity and surface tension become important. Here we neglect these, as well as evaporation.
[11] Assume that p_a is constant.

a. The volume of water in the bucket is $A\,h$, and the flow rate out of it is $a\,v$. Hence, conservation of water gives $\boldsymbol{A\,\dot{h} = -a\,v}$.

b. **1.** The potential energy of the water in the bucket is $V = \dfrac{1}{2}\,g\,A\,\rho\,h^2$.

2. The kinetic energy flux is $K_f = \dfrac{1}{2}\,\dot{m}\,v^2$, where $\dot{m} = -\rho\,A\,\dot{h}$ is the rate at which mass leaves the bucket. Hence $K_f = -\dfrac{1}{2}\,\rho\,A\,v^2\,\dot{h}$.

From conservation of energy $\dot{V} + K_f = 0$, so that .... $\boldsymbol{v^2 = 2\,g\,h}$.

Note: an alternative expression for the kinetic energy flux is $K_f = \dfrac{1}{2}\,\rho\,a\,v^3$. This because, in an infinitesimal time interval dt, the kinetic energy transported out is $dK = \dfrac{1}{2}\,(\rho\,a\,v\,dt)\,v^2$. From the result in **a**, this is equivalent to the expression in **2** above.

c. If the pressure is hydrostatic, then $p = g\,\rho\,h + p_a$ at the bottom of the bucket. Thus Bernoulli's law reduces to $\frac{1}{2}\,\rho\,v^2 = p - p_a = g\,\rho\,h$, which gives the same result as **b**.

Equation (7.1) follows from **a** and **b** above, upon use of the fact that both h and v are non-negative.

8 Well and/or ill posed PDE #02

8.1 Statement: Well and/or ill posed PDE #02

Let $\rho = \rho(x,\,t)$ and $q = q(x,\,t)$ be the density (and corresponding flux) for some conserved quantity. Then, in the absence of sources:

$$\rho_t + q_x = 0, \quad -\infty < x < \infty. \tag{8.1}$$

Assume that, while examining the physical problem leading to this equation, you conclude that a "good approximation" for the flux is

$$q = \rho + c\,\rho_{xxx}, \tag{8.2}$$

where c is some constant. Substituting (8.2) into (8.1) yields a pde for ρ. *What restriction should you impose on the constant c so that this pde is not ill-posed?* Specifically: what restriction on c guarantees that the equation does not exhibit arbitrarily large growth factors at high frequencies?

Hint. *Examine the behavior of sinusoidal in space solutions: $\rho \propto e^{i\,k\,x}$ with k a real constant.*

8.2 Answer: Well and/or ill posed PDE #02

The pde for ρ resulting from (8.1 – 8.2) is

$$\rho_t + \rho_x + c\,\rho_{xxxx} = 0. \tag{8.3}$$

It is easy to check that $\rho = \exp(i\,k\,x + \lambda\,t)$ is a solution of this equation if and only if

$$\lambda = -i\,k - c\,k^4. \tag{8.4}$$

Hence, to avoid unbounded growth as $|k| \to \infty$, **we must require $\boldsymbol{c \geq 0}$.**

Remark 8.1 *Equation (8.2) is a made-up flux. I do not know of a physical problem where this happens. This does not mean that there is none, only that I do not know of it.*

9 Well and/or ill posed PDE #05

9.1 Statement: Well and/or ill posed PDE #05

Let $\rho = \rho(x,\, t)$ and $q = q(x,\, t)$ be the density (and corresponding flux) for some conserved quantity. Then, in the absence of sources:

$$\rho_t + q_x = 0, \quad -\infty < x < \infty. \tag{9.1}$$

While examining the physical problem leading to this equation, you conclude that "good approximations" for the density and flux are

$$\rho = u_t \quad \text{and} \quad q = u_x - c\, u_{x\,x\,x} - 2\, u_{xt}, \tag{9.2}$$

where c is a constant, and $u = u(x,\, t)$. Substituting (9.2) into (9.1) yields a pde for u. *What restriction should you impose on c so that this pde is not ill-posed?* Specifically: what restriction guarantees that the equation does not exhibit arbitrarily large growth factors at high frequencies?

Hint. *Examine the behavior of sinusoidal in space solutions: $u \propto e^{i\,k\,x}$ with k a real constant.*

Warning: *To have an ill-posed situation, arbitrarily large growth factors are needed. Growth alone, if bounded, is not enough — this is a sign of some instability, not the same as being ill-posed.*

9.2 Answer: Well and/or ill posed PDE #05

The pde for u resulting from (9.1 – 9.2) is

$$u_{tt} + u_{xx} - 2\, u_{xxt} - c\, u_{xxxx} = 0. \tag{9.3}$$

It is easy to check that $\rho = \exp(i\,k\,x + \lambda\,t)$ is a solution of this equation if and only if

$$\lambda = -k^2 \pm \sqrt{k^2 + (1+c)\,k^4}. \tag{9.4}$$

Let us now examine the behavior of λ for $|k|$ large. The following cases arise

1. If $(1+c) > 0$, $\lambda = \left(-1 \pm \sqrt{1+c}\right)\, k^2 + O(1)$. Unbounded growth rates result if $1 + c > 1$.
2. If $(1+c) = 0$, $\lambda = -k^2 \pm k$. The growth rate is bounded. The maximum possible occurs for $k = \pm\frac{1}{2}$, where $\lambda = \frac{1}{4}$ is possible. Furthermore, for $|k| > 1$, $\mathrm{re}(\lambda) < 0$.
3. If $(1+c) < 0$, $\lambda = \left(-1 \pm i\,\sqrt{|1+c|}\right)\, k^2 + O(1)$. The growth rate is bounded, with $\mathrm{re}(\lambda) < 0$ for all $k \neq 0$.

Thus, to avoid unbounded growth as $|k| \to \infty$, **we must require $c \leq 0$.**

Remark 9.1 *When $0 \leq 1 + c \leq 1$, the equation is well-posed, but the wave-lengths satisfying $0 < k^2 < -\frac{1}{c}$ can grow exponentially (one of the two λ is positive). However, the growth rate is bounded.*

Remark 9.2 *The density and flux in (9.2) are made up. I do not know of a physical problem where this happens (this does not mean that there is none, only that I do not know of it).*

Here we calculate the maximum growth rate for the case where $0 < 1 + c = \mu^2 < 1$. For this purpose we need to find the maximum of $\lambda = -k^2 + \sqrt{k^2 + \mu^2\,k^4}$ in the interval $0 < k^2 < 1/(1-\mu^2)$.

Let $\boldsymbol{z = \mu\, k^2 + \frac{1}{2\,\mu}}$. Then $\boldsymbol{\lambda = \frac{1}{2\,\mu^2} - \frac{z}{\mu} + \sqrt{z^2 - \frac{1}{4\,\mu^2}}}$ and $\boldsymbol{1 < 2\,\mu\, z < \frac{1+\mu^2}{1-\mu^2}}$.

It is then easy to see that the maximum, $\boldsymbol{\lambda = \frac{1-\sqrt{1-\mu^2}}{2\,\mu^2}}$, happens at $\boldsymbol{2\,\mu\,z = \frac{1}{\sqrt{1-\mu^2}}}$.

THE END.