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1 Nonlinear solvable ODEs

1.1 Statement: A nonlinear solvable ode #02

Generally nonlinear ode, even simple scalar and first order ones, do not admit explicit exact solutions. There are, however, some exceptions. For example, consider the ode

\[ 0 = x + \frac{1 - y}{y'} + \left( x - \frac{y}{y'} \right)^2, \tag{1.1} \]

where \( y' = \frac{dy}{dx} \). This ode is nonlinear with variable coefficients, and not separable. Yet it can be solved exactly, even though the standard methods do not apply. Your task here is to do this.

**Hint.** Introduce the variables,

\[ u = \frac{1}{y'} \quad \text{and} \quad v = x - \frac{y}{y'} = x - y, \tag{1.2} \]

and investigate what equation (1.1) implies for them.

1.2 Answer: A nonlinear solvable ode #02

From (1.1) and (1.2) it follows that

\[ 0 = v + u + u^2. \tag{1.3} \]

Furthermore

\[ \frac{dv}{dx} = -y \frac{du}{dx} = y \left( 1 + 2 v \right) \frac{dv}{dx}, \tag{1.4} \]

where the first equality follows from the definition of \( u \) and \( v \), and the second upon use of (1.3). It follows that, either \( dv/dx = 0 \) or \( y(1 + 2v) = 1 \). We analyze these two cases below.

**The case \( dv/dx = 0 \).**

In this case \( v = x_0 \) is a constant. Using this fact in the definition of \( v \) in (1.2) yields

\[ \frac{y'}{y} = \frac{1}{x - x_0} \implies y = c(x - x_0), \tag{1.5} \]

where \( c \) is some constant. Substituting this back into equation (1.1) yields

\[ 0 = x_0 + \frac{1}{c} + x_0^2. \tag{1.6} \]

Hence we arrive at the family of solutions to (1.1) given by

\[ y = -\frac{x - x_0}{x_0 + x_0^2}, \tag{1.7} \]

where \( x_0 \) is an arbitrary constant — with the restriction that \( x_0 \neq 0 \) and \( x_0 \neq -1 \).
The case \( y(1 + 2v) = 1 \).

Then \( v = \frac{1}{2y} - \frac{1}{2} \). Thus, from (1.3), \( u = \frac{1}{4} - \frac{1}{4y^2} \). But, from (1.2), \( v = x - yu \). Hence

\[
\frac{1}{2y} - \frac{1}{2} = x - y \left( \frac{1}{4} - \frac{1}{4y^2} \right) \iff 0 = y^2 - (4x + 2)y + 1.
\]

We conclude that

\[
y = 1 + 2x \pm 2 \sqrt{x + x^2},
\]

where \( y \) is real valued for \( x \leq -1 \) or \( x \geq 0 \). By direct substitution, it is easy to check that these are solutions of (1.1).

## 2 Linear 1st order PDE (problem 04)

### 2.1 Statement: Linear 1st order PDE (problem 04)

Discuss the two problems

\[
u_x + 2xu_y = y, \quad \text{with} \quad \begin{cases} \text{(a)} & u(x, x^2) = 1 \quad \text{for} \quad -1 < x < 1, \\ \text{(b)} & u(x, x^2) = \frac{1}{3}x^3 + \pi \quad \text{for} \quad -1 < x < 1. \end{cases}
\]

How many solutions exist in each case?

**Note:** the data in these problems is prescribed along a characteristic!

### 2.2 Answer: Linear 1st order PDE (problem 04)

Using \( x \) to parametrize the characteristic equations for the problem in (2.1), we obtain

\[
dy/dx = 2x \quad \text{and} \quad du/dx = y.
\]

These equations have the general solution

\[
y = \tau + x^2 \quad \text{and} \quad u = \mu + \tau x + \frac{1}{3}x^3,
\]

where \( \tau \) and \( \mu \) are arbitrary constants. For \( \tau = 0 \) this yields \( u(x, x^2) = \mu + \frac{1}{3}x^3 \), which is inconsistent with (a) in (2.1), and consistent with (b) in (2.1) — provided that \( \mu = \pi \).

**Part 1.** Problem (a) in (2.1) has no solutions.

**Part 2.** Problem (b) in (2.1) has infinitely many solutions. These are obtained by first using the first expression in (2.3) to obtain \( \tau = y - x^2 \), and then taking \( \mu \) as a function of \( \tau \) in the second expression (with \( \mu = \pi \) for \( \tau = 0 \)). In other words

\[
u = f(y - x^2) + x(y - x^2) + \frac{1}{3}x^3,
\]

where \( f \) is any function such that \( f(0) = \pi \).
3 Linear 1st order PDE (problem 08)

3.1 Statement: Linear 1st order PDE (problem 08)

Solve the problem below, using the method of characteristics: (a) Compute the characteristics, as done in the lectures, starting from each point in the data set. (b) Next solve for the solution $u$ along each characteristic. (c) Finally, eliminate the characteristic variables $\zeta$ and $s$ from the expression for $u$ obtained in step (b) — using the result in step (a) — to obtain the solution as a function of $x$ and $y$.

$$(x - y)u_x + (x + y)u_y = x^2 + y^2,$$  \hfill (3.1)

with data $u(x, 0) = (1/2)x^2$ for $1 \leq x < \exp(2\pi)$.

**Answer this question:** Where in the $(x, y)$ plane does the problem above define the solution $u$? That is: what is the region of the plane characterized by the property that: through each point in this region there is exactly one characteristic connecting it with the curve where the data is given?

3.2 Answer: Linear 1st order PDE (problem 08)

The characteristic form of equation (3.1) is

$$\frac{dx}{ds} = x - y, \quad \frac{dy}{ds} = x + y, \quad \text{and} \quad \frac{du}{ds} = x^2 + y^2.$$  \hfill (3.2)

The given data then translates into the conditions

$$x = \zeta, \quad y = 0, \quad \text{and} \quad u = \frac{1}{2} \zeta^2,$$  \hfill (3.3)

for $1 \leq \zeta < e^{2\pi}$ and $s = 0$.

The equations in (3.2 — 3.3) have the solution

$$x = \zeta e^s \cos s, \quad y = \zeta e^s \sin s, \quad \text{and} \quad u = \frac{1}{2} \zeta^2 e^{2s}.$$  \hfill (3.4)

Clearly, for any fixed $\zeta$, the curves in the plane that these formulas define are counter-clockwise logarithmic spirals, scaling by a factor $e^{2\pi}$ per cycle. Since $1 \leq \zeta < e^{2\pi}$, these curves cover the whole plane (except the origin), with exactly one curve going through each point such that $x^2 + y^2 > 0$.

From (3.4), it follows easily that the solution to (3.1) is

$$u = \frac{1}{2} \left( x^2 + y^2 \right),$$  \hfill (3.5)

which is defined (and unique) on $0 < x^2 + y^2 < \infty$.

**Remark 3.1** Notice that (3.5) has a unique limit as $x^2 + y^2 \to 0$. However, this is not generic. For example, consider the case when the data is $u(x, 0) = 0$ for $1 \leq x < e^{2\pi}$. Then $u = \frac{1}{2} \zeta^2 \left( e^{2s} - 1 \right)$. But $x^2 + y^2 \to 0$ corresponds to $s \to -\infty$ along characteristics. Hence $u$ has no limit as $x^2 + y^2 \to 0$.

---

1 Here $\zeta$ is the label for each characteristic, and $s$ is a parameter along the characteristics.
4 Semi-Linear 1st order PDE (problem 04)

4.1 Statement: Semi-Linear 1st order PDE (problem 04)

Consider the following problem, in the region $x, t > 0$,
\[ u_t + \frac{1+x}{1+t} u_x = -u^2, \quad \text{with data} \quad u(x, 0) = \frac{1}{1+x} \quad \text{and} \quad u(0, t) = \frac{1}{1+t}. \quad (4.1) \]

1. Use the method of characteristics to solve the problem. Write explicit formula(s) for $u$ for all the values $x, t > 0$ for which the solution is defined. Describe the domain of definition $\Omega$.

2. Use the result in 1 to show that the solution is smooth (infinite partial derivatives) in the region $\Omega$, except along a special curve $\Gamma$ — describe $\Gamma$.

3. Is the solution continuous for points on $\Gamma$? What happens with $u_t$ and $u_x$ for points on $\Gamma$?

4. For the solution $u = u(x, t)$ found in item 1, does the pde\(^2\) in (4.1) make sense for points on $\Gamma$? In other words, does $u$ satisfy the pde everywhere in $x, t > 0$?

4.2 Answer: Semi-Linear 1st order PDE (problem 04)

The characteristics for (4.1) are the curves satisfying the ode $\dot{x} = (1 + x)/(1 + t)$. Thus they have the form $x = -1 + c_1(1 + t)$, where $c_1$ is a constant. Along each characteristic, $u$ satisfies $\dot{u} = -u^2$, hence $u = 1/(c_2 + t)$ — where $c_2$ is another constant. Inspecting the characteristics starting along the boundary of the region $x, t > 0$, we obtain:\(^3\)
\[ u = \frac{1}{1 + \xi + t} \quad \text{along} \quad x = -1 + (\xi + 1)(1 + t), \quad \text{for} \quad \xi \geq 0 \text{ and } t \geq 0, \quad (4.2) \]
and
\[ u = \frac{1}{1 + t} \quad \text{along} \quad x = -1 + \frac{1 + t}{1 + \tau}, \quad \text{for} \quad \tau \geq 0 \text{ and } t \geq \tau. \quad (4.3) \]

Note that the characteristics corresponding to $\xi = 0$, and that for $\tau = 0$, are the same.

Solving for $\xi$ as a function of $t$ and $x$ in (4.2) yields the formula
\[ u = \frac{1 + t}{1 + x + t + t^2} \quad \text{for} \quad x \geq t. \quad (4.4) \]
Similarly, (4.3) produces
\[ u = \frac{1}{1 + t} \quad \text{for} \quad x \leq t. \quad (4.5) \]

We conclude that

1. The solution is given by (4.4) and (4.5), in the region $x, t > 0$ — in particular: $\Omega$ is the whole region.

2. The solution is smooth at all the points in the region $x, t > 0$ satisfying $x \neq t$. The curve $\Gamma$ is given by $x = t$.

3. The solution is continuous everywhere, even along $\Gamma$. However, the partial derivatives $u_t$ and $u_x$, while continuous for either $x \geq t$ or $x \leq t$, have jumps across $\Gamma$. Hence, if we were to plot the surface $u = u(x, t)$ in space time, we would see that it has a ridge (like the crest of a mountain ridge) along $\Gamma$.

4. While $u_t$ and $u_x$ are discontinuous across $\Gamma$, the expression $u_t + \frac{1+x}{1+t} u_x + u^2$ is continuous everywhere (and identically zero). Hence the pde in (4.1) makes sense everywhere, even on $\Gamma$.

---

\(^2\)That is: $u_t + \frac{1+x}{1+t} u_x = -u^2$.

\(^3\)To understand what is going on, I strongly recommend that you draw the characteristic curves.
5 Fundamental Diagram of Traffic Flow #01

5.1 Statement: Fundamental Diagram of Traffic Flow #01

The desired car velocity $u = U(\rho)$ has its maximum, $u_m$, at $\rho = 0$, and vanishes at the jamming density, $\rho_J$. Assuming that $U$ is a linear function of $\rho$, write a formula for the flow rate $q = Q(\rho)$. What is the road capacity $q_m$? What is the wave velocity $c = c(\rho)$? 

*Note:* The road capacity is the maximum flow rate that a road allows.

5.2 Answer: Fundamental Diagram of Traffic Flow #01

From the stated assumptions

$$U = u_m \left(1 - \frac{\rho}{\rho_J}\right).$$

(5.1)

Hence, from $q = \rho u$, it follows that

$$Q = u_m \rho \left(1 - \frac{\rho}{\rho_J}\right) \implies q_m = \frac{1}{2} \rho_J u_m.$$  

(5.2)

Furthermore

$$c = \frac{dq}{d\rho} = u_m \left(1 - 2 \frac{\rho}{\rho_J}\right).$$

(5.3)

6 Fundamental Diagram of Traffic Flow #03

6.1 Statement: Fundamental Diagram of Traffic Flow #03

Many state laws state that: for each 10 mph (16 kph) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law “literally” (i.e. they use exactly one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 ft (5 m)). There is another law that gives a maximum speed limit (assume that this is 50 mph (80 kph)). *Find the flow of cars as a function of density, $q = q(\rho)$, that results from these two laws.*

The state laws on following distances stated in the prior paragraph were developed in order to prescribe a spacing between cars such that rear-end collisions could be avoided, as follows:

- **a.** Assume that a car stops instantaneously. *How far would the car following it travel* if moving at $u$ mph and
  
  **a1.** The driver’s reaction time is $\tau$, and
  
  **a2.** After a delay $\tau$, the car slows down at a constant maximum deceleration $\alpha$.

- **b.** The calculation in part a may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate $\alpha$, but the driver in the following car still takes a time $\tau$ to react. *How far back does a car have to be, traveling at $u$ mph, in order to prevent a rear-end collision?*

- **c.** Show that the law described in the first paragraph of this problem corresponds to part b, if the human reaction time is about 1 sec. and the length of a car is about 16 ft (5 m). 

*Note:* What part c is asking you to do is to justify/derive the state law prescription, using the calculations in part b to arrive at the minimum car-to-car separation needed to avoid a collision when the cars are forced to brake.
6.2 Answer: Fundamental Diagram of Traffic Flow #03

Assume that the drivers follow the state law prescription exactly. Then

\[ d = \frac{L}{V}, \]  

(6.1)

where \( d \) is the distance to the next car, \( L \) is the car length, \( u \) is the car velocity and \( V \) is the law “trigger” velocity, as in:

**State Law:** Maintain a distance of one car length for each \( V \) increase in velocity.

Typical numbers are \( V = 10 \text{ mph} = 16 \text{ kph} \) and \( L = 16 \text{ ft} = 5 \text{ m} \).

Since we then end up with one car for every \( d + L \) distance, equation (6.1) above leads to the velocity–density relationship

\[ \frac{1}{\rho} = L + d = L \left( 1 + \frac{u}{V} \right) \quad \text{or} \quad u = \left( \frac{1}{\rho L} - 1 \right) V. \]  

(6.2)

Thus the car flux is given by

\[ q = u \rho = \left( \frac{1}{L} - \rho \right) V \quad \Rightarrow \quad c = \frac{dq}{d\rho} = -V. \]  

(6.3)

Hence this flux gives a constant wave speed \( c \), and has a car velocity that goes to infinity as the density vanishes. This is because we have not yet enforced the **speed limit restriction**, which yields

\[ u = \min \left\{ u_m, \left( \frac{1}{\rho L} - 1 \right) V \right\} \quad \text{and} \quad q = \min \left\{ \rho u_m, \left( \frac{1}{L} - \rho \right) V \right\}, \]  

(6.4)

where \( u_m \) = speed limit. The **critical density** below which \( u = u_m \), is

\[ \rho_c = \frac{V}{L(V + u_M)}. \]  

(6.5)

Then (6.2 – 6.3) applies for \( \rho \geq \rho_c \), or \( u \leq u_m \).

Next we motivate the state laws by a simple calculation involving two facts: (i) Drivers have a finite reaction time, \( \tau > 0 \). (ii) Cars do not change velocity instantaneously, but do so with a finite deceleration \( \alpha > 0 \). To simplify matters, here we assume that \( \tau \) is the same for all the drivers, and that \( \alpha \) is constant (and the same for all cars).

Imagine now two cars, one behind the other, traveling at the same (constant) speed \( u \). At some point, the car ahead (car #1) starts braking. It then travels a distance

\[ D_1 = \frac{1}{2} \frac{u^2}{\alpha}. \]  

(6.6)

from the moment the brakes are applied to the moment it stops. On the other hand, the distance traveled by the car behind (car #2) from the moment the driver sees that he must stop \(^4\) till the car actually stops, includes the driver’s reaction time. That is

\[ D_2 = u \tau + \frac{1}{2} \frac{u^2}{\alpha}. \]  

(6.7)

It then follows that, in order to avoid a rear end collision, the distance between two cars traveling at speed \( u \) must be at least \( u \tau \). This yields the following formula for the “trigger” velocity

\[ V = \frac{L}{\tau}. \]  

(6.8)

In particular, \( L = 16 \text{ ft} \) and \( \tau = 1 \text{ s} \) yields \( V = 16 \text{ ft/s} = 10.9 \text{ mph} \) — note that 1 mile = 5280 ft and 1 hr = 3600 s.

This trigger velocity is pretty close to the one used in many state laws.

**Note:** The calculation above is a bit sloopy, for it only checks that car #2 is still behind car #1 once they stop. What one should check is that car #2 stays behind car #1 at all times. However car #2 is always moving at a speed equal to or greater than that of car #1. This because it starts slowing down, at the same rate and starting from the same speed, later. Thus the distance between the two cars is a non-increasing function of time. It follows that it is enough to check that it is positive once they stop, to know that it was always positive.

\(^4\) Say, the brake lights for the car ahead turn on.
7 Envelope of the characteristics and cusp

7.1 Statement: Envelope of the characteristics and cusp

Consider the problem

\[ c_t + c c_x = 0, \quad \text{where} \quad c(x, 0) = C(x), \quad \text{for} \quad -\infty < x < \infty. \]  \hfill (7.1)

**e1.** Write the characteristic curves for this problem, each one parameterized by time, and labeled by the value of \( x = s \) at time \( t = 0 \). That is, write formulas for the characteristics of the form \( x = X(s, t) \).

**e2.** Write the equation for the envelope of the characteristics, and express the envelope in parametric form \( x = X_e(s) \) and \( t = T_e(s) \).

**e3.** Assume that \( C \) has an inflection point at \( x = 0 \). In fact, assume that \( C(0) = 0, C'(0) = -a < 0, C''(0) = 0, \) and \( C'''(0) = 2ab > 0 \), \hfill (7.2)

where the primes denote derivatives.

Let \( x_c = X_e(0) \) and \( t_c = T_e(0) \). **Show that** \( T_e \) **has a local minimum at** \( s = 0 \), and that the envelope has a **cusp** at \( (x_c, t_c) \).

**HINT:** Expand the equations for \( s \) small.

**e4.** Replace (7.1) by \[ c_t + c c_x = -c, \quad \text{where} \quad c(x, 0) = C(x), \quad \text{for} \quad -\infty < x < \infty. \] \hfill (7.3)

**Repeat steps** e1 and e2 **for this problem. Question:** what condition is needed on \( C' \) so that the characteristics of (7.3) actually **have** an envelope in the upper half space-time plane? That is, so that \( T_e(s) > 0 \) somewhere.

**Note:** Assume a one parameter set of curves in the plane, defined by an equation of the form \( F(x, t, s) = 0 \), where \( s \) is the parameter. Then the envelope is the set of points in the plane which are the intersections of two “infinitesimally close” curves. That is, points that satisfy both \( F(x, t, s) = 0 \) and \( F(x, t, s + ds) = 0 \).

Expanding the second equation we see that the **envelope is defined by** \( F(x, t, s) = 0 \) and \( F_s(x, t, s) = 0 \). \hfill [A]

Given the definition above of the envelope, it should be clear that the **envelope of characteristics is precisely the place where the derivatives of the solution by characteristics of (7.1) become infinity:** at \( t = 0, \) \( dx = ds \), and there is some \( dc \) given by the initial conditions that result in a finite \( \frac{dc}{ds} \). However, at the envelope \( dc \) is still the same, while \( dx = 0; \) hence \( e_s = \pm \infty \).

An **alternative definition** of the envelope is: it is a curve \( x = X(s) \) and \( y = Y(s) \) such that the point \( (X(s), Y(s)) \) in the curve is also in the curve with label \( s \) belonging to the family; and both curves are tangent there.\(^5\) What this means is that the “neighboring” characteristics that intersect at the envelope, are also tangent to the envelope there.

7.2 Answer: Envelope of the characteristics and cusp

**e1.** The characteristic curves for the problem in (7.1) are given by

\[ x = C(s) t + s = X(s, t), \quad \text{where} \quad -\infty < s < \infty \quad \text{and} \quad t > 0. \] \hfill (7.4)

**e2.** The envelope of the curves in (7.4) is given by the two equations

\[ x = C(s) t + s = X(s, t) \quad \text{and} \quad 0 = C'(s) t + 1. \] \hfill (7.5)

Hence

\[ t = -\frac{1}{C'(s)} = T_e(s) \quad \text{and} \quad x = s - \frac{C(s)}{C'(s)} = X_e(s). \] \hfill (7.6)

\(^5\) You can check that this also leads to the equations in [A].
e3. Expand \( C = -a \left( s - \frac{1}{3} b s^3 \right) + O(s^4) \) and \( C' = -a (1 - b s^2) + O(s^3) \). Thus

\[
T_e = \frac{1}{a} \left( 1 + b s^2 \right) + O(s^3) \quad \text{and} \quad X_e = -\frac{2}{3} b s^3 + O(s^4).
\] (7.7)

Clearly \( T_e \) has a local minimum at \( s = 0 \). Further, with \( t_c = 1/a \) and \( x_c = 0 \), we see that on the envelope

\[
(t - t_c) = \left( \frac{3^{2/3} b^{1/3}}{2^{2/3} a} \right) \left( (x - x_c)^2 \right)^{1/3} + O(x - x_c).
\] (7.8)

Hence the envelope has a cusp at \((x_c, t_c)\).

e4. The characteristics for (7.3) are given by \(6\)

\[
x = C(s) \left( 1 - e^{-t} \right) + s = X(s, t), \quad \text{where} \quad -\infty < s < \infty \quad \text{and} \quad t > 0.
\] (7.9)

The envelope of these curves is given by the equations

\[
x = C(s) \left( 1 - e^{-t} \right) + s \quad \text{and} \quad 0 = C'(s) \left( 1 - e^{-t} \right) + 1.
\] (7.10)

The second equation has a solution with \( t > 0 \) if and only if \( C'(s) < -1 \) somewhere. Given this we can write

\[
T_e(s) = -\ln \left( 1 + \frac{1}{C'(s)} \right) \quad \text{and} \quad X_e(s) = s - \frac{C(s)}{C'(s)}.
\] (7.11)

8 Small vibrations of a 2D string under tension (sv2ds)

8.1 Introduction: Small vibrations of a 2D string under tension

Here you are asked to derive equations for the transversal vibrations of a thin elastic string under tension, under several scenarios. The following hypotheses are common to all of them (8.1)

1. The string is homogeneous, with mass density \( \rho = \text{constant} \).
2. The motion is restricted to the \( x-y \) plane. At equilibrium the string is described by \( y = 0 \) and \( 0 \leq x \leq L = \text{string length} \). The tension \( T > 0 \) is then constant. See remark 8.1, #1–3.
3. The string has no bending strength.
4. The amplitude of the vibrations is very small compared with their wavelength, and any longitudinal motion can be neglected. Thus:
   - The string can be described in terms of a deformation function \( u = u(x, t) \), such that the equation for the curve describing the string is \( y = u(x, t) \).
   - The tension remains constant throughout — see remark 8.1, #4.

Remark 8.1 Some details:

#1 We idealize the string as a curve. This is justified as long as \( \lambda \gg d \), where \( \lambda = \text{scale over which motion occurs} \), and \( d = \text{string diameter} \). This condition justifies item 3 as well.

6 Along each characteristic \( c = C(s) e^{-t} \).

7 In this approximation the \( x \)-coordinate of any mass point on the string does not change in time.
#2 The tension is generated by the elastic forces (assume that the string is stretched). At a point along the string, the tension is the force with which each side (to the right or left of the point) pulls on the other side,\(^8\) and it is directed along the direction tangent to the string (there are no normal forces, see item 3).

#3 At equilibrium the tension must be constant. For imagine that the tension is different at two points \(a < b\) along the string. Then the segment \(a \leq x \leq b\) would receive a net horizontal force (the difference in the tension values), and thus could not be at equilibrium.

#4 The hypotheses imply that string length changes can be neglected. Thus the stretching generating the tension does not change significantly, and the tension remains equal to the equilibrium tension \(T\).

8.2 Statement: sv2ds01 (no other forces/effects)

Here we consider the simplest situation, where there is no other force beside the tension \(T\) on the string, nor any dissipation. Under these conditions, plus those in § 8.1:

Use the conservation of the transversal momentum to derive an equation for \(u\). Thus:

i) Obtain a formula for the transversal momentum density along the string (momentum per unit length).

ii) Obtain a formula for the transversal momentum flux along the string (momentum per unit time).

iii) Use the differential form of the conservation of transversal momentum to write a pde for \(u\).

Hint. Remember that internal string forces\(^9\) are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces. Be careful with the sign here! Remember also that \(u_x\) is very small.

The pde for \(u\) derived above applies for \(0 < x < L\). For a solution to this equation to be determined, extra conditions on the solution \(u\) at each end are needed. These are called boundary conditions, and must be derivedmodeled as well. Later on we will see that exactly one boundary condition (a restriction on \(u\) and its derivatives) is needed at each end, \(x = 0\) and \(x = L\).

Answer the following extra questions:

iv) What boundary condition applies at an end where the string is tied? (e.g., as in a guitar).

v) What boundary condition applies at an end where the string is free?

For example, imagine a no-mass cart that can slide up and down (without friction) a vertical rod at \(x = 0\), and tie the left end of the string to this cart. This keeps the left end of the string at \(x = 0\), with the rod canceling the horizontal component of the tension force, but \(u(0, t)\) can change. Note that a no-mass-no-friction cart is an idealization, approximating the set-up in (vi) in the limit where the mass and the friction are small.

vi) What boundary condition applies at an end where there is a mass \((M)\) attached to the string, with a force \(f = -bf v_m\) opposing the mass motion (here \(bf = \) constant and \(v_m\) = mass velocity).

An example of this would be the cart in (v), but neglecting neither the cart mass, nor the friction as it moves up or down along the vertical rod.

Warning: the form of boundary condition in this case depends on which end of the string the bead is on. This is because the force by the string tension on the bead has a sign change (why?)

vii) Let \(\lambda\) be a typical wavelength for the string vibrations. Then \(\tau = \lambda/c\), where \(c = \sqrt{T/\rho}\), is the typical time scale (why?). Under which conditions on \(\rho\), \(T\), \(M\), \(bf\), and \(\lambda\) is the free end boundary condition in (v) a good approximation to the boundary condition in (vi)?

Hint. Let \(a\) denote a typical amplitude for the string vibrations. Then we expect that \(u_t = O(a/\tau)\), \(u_x = O(a/\lambda)\), and so on. Use this to compare the size of the terms in the boundary condition derived in (vi). From this you should be able to obtain two a-dimensional numbers that must be small for the boundary condition in (vi) to reduce to the one in (v). One of them measures the ratio of the bead inertial forces to the string tension, and the other the ratio of the friction forces to the string tension.

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\(^8\) If you were to cut the string, this would be the force needed to keep the lips of the cut from separating.

\(^9\) Internal forces = forces by one part of the string on another.
8.3 Answer: sv2ds01

We have

1) The transversal momentum per unit length along the string is given by

\[ \rho_{\text{mom}} = \rho u_t. \]

2) Momentum flux along the string happens because of transversal forces.

That is: the force in the \( y \)-direction that, at every point, one side of the string applies on the other side. This is just the \( y \)-component of the tension, which is given by \( T \sin \theta \), where \( \theta \) is the angle that the string tangent makes with the \( x \)-axis. However, since \( u_x \) is small, \( \theta \approx u_x \). Thus the transversal momentum flux is given by

\[ q_{\text{mom}} = -Tu_x. \]

Note about the sign here: when \( u_x < 0 \), the left side of the string pulls the right side up, generating a positive momentum flux. Vice-versa, when \( u_x > 0 \), momentum flows from right to left (negative). Thus the sign is as above.

Note also that the \( x \)-component of the tension (horizontal force along the string) is \( T \cos \theta = T \), since \( u_x \) is small. Thus it is constant, consistent with the approximation of no motion in the \( x \) direction.

3) From conservation \( (\rho_{\text{mom}})_t + (q_{\text{mom}})_x = 0 \). That is

\[ \rho u_{tt} - Tu_{xx} = 0. \] (8.2)

Alternatively

\[ u_{tt} - c^2 u_{xx} = 0, \] (8.3)

where \( c^2 = T/\rho \) is a speed (the wave speed).

4) Tied end. Suppose the left end of the string is tied. Then its position is prescribed there (say: \( y = a \) at \( x = 0 \)), which leads to

\[ u(0, t) = a, \] (8.4)

where \( a \) is a constant. More generally, we can have \( u(0, t) = \sigma(t) \).

5) Free end. Suppose the string’s left end is free, so there is no force there to balance any transversal component of the tension. Hence

\[ u_x(0, t) = 0. \] (8.5)

6) Mass at the left end of the string. Here we assume that the string is attached to a bead which slides along a straight rod, with friction producing a force proportional to the bead velocity along the rod (and opposing its motion). The rod is positioned so that the bead moves along the line \( x = 0, -\infty < y < \infty \).

Let \( M \) and \( b_f > 0 \) be the bead mass and friction coefficient. Then, since the bead position is given by \( y = u(0, t) \), we have

\[ M u_{tt}(0, t) = -b_f u_t(0, t) + Tu_x(0, t). \] (8.6)

Mass at the right end of the string. Then the string pulling on the mass is to the right (instead of to the left). Thus we must reverse the sign of the tension force on the bead, and obtain

\[ M u_{tt}(0, t) = -b_f u_t(0, t) - Tu_x(0, t). \] (8.7)

7) The boundary condition \( u_x = 0 \) is a good approximation to (8.6–8.7) provided that \( \frac{T}{X} \gg \frac{M}{M} \) and \( \frac{X}{X} \gg \frac{b_f}{b_f} \).

That is, provided that the two a-dimensional numbers

\[ \frac{M \lambda}{T \tau^2} = \frac{M c^2}{T \lambda} = \frac{M}{\lambda \rho} \]

and

\[ \frac{b_f \lambda}{X \tau} = \frac{b_f c}{T} = \frac{b_f}{\sqrt{\rho T}} \]

are small. (8.8)

Why is \( \tau = \lambda/c \)? From the equation \( u_{tt} = c^2 u_{xx} \), where the balance of the two sides yields \( c^2 \tau^2 \sim \lambda^2 \).

8.3.1 Conservation of energy

In the derivation above we obtained an equation for the string using, solely, the conservation of the transversal momentum. What happens with the other quantities that are conserved? They should be conserved. Let us check that this is so.
Conservation of mass. This is guaranteed by the parameterization we use. That is, each point \( (x, u) \) follows a specific bit of the spring, with mass \( \rho dx \). Mass is automatically conserved. This is typical of solid mechanics problems, where the parameterization tracks points in the object. In fluids individual particles are not tracked; instead the flow velocity at each point in space is prescribed. Thus mass conservation is not automatic, and must be enforced. On the other hand, in fluids the equations involve only first time derivatives of the densities — while in solids second order equations occur. This is because the velocities are variables in fluid problems, while in solids they result from the time derivatives of the displacements.

Conservation of longitudinal momentum. Since there is no motion in the \( x \)-direction, and the horizontal component of the tension forces is constant (see the end of item (ii) of the answer), the longitudinal momentum is conserved as well (trivially so). In fact, the statement that the derivation above does not use the conservation of the longitudinal momentum is not quite correct! This is used, effectively, when concluding that the tension on the string is constant.

Conservation of energy. To show that energy is conserved, we first need to calculate its density and its flux. The energy density has two components: kinetic energy per unit length \( \frac{1}{2} \rho u_t^2 \), and the potential energy stored in the stretching of the string, which we need to compute. In fact we only need to compute the difference between the potential energy at any time, and some constant energy — specifically: the potential energy of the string at equilibrium. But then the tension is constant through the whole stretching process taking the string from one state to the other, so that the energy is the product of \( T \) times the change in length. This yields potential energy per unit length \( T (\sqrt{1 + u_x^2} - 1) = \frac{1}{2} T u_x^2 \), since the arc-length is \( ds = \sqrt{1 + u_x^2} dx \), and \( u_x \) is small. Finally, the flux of energy is given by the work (per unit time) done by the transversal force (which we already calculated in (ii) of the answer). That is: energy flux \( -T u_x u_t \). It follows that the equation for the conservation of energy is

\[
\left( \frac{1}{2} \rho u_t^2 + \frac{1}{2} T u_x^2 \right)_t - (T u_x u_t)_x = 0.
\]

(8.9)

It is easy to see that (8.2) guarantees this.

This calculation does not involve an internal energy, because in this model there is no mechanism for energy exchanges between mechanical and internal. This is the reason why energy conservation is “automatic”.

Note: here we present an alternative (perhaps more intuitive) way to get the formula \( V_e = \frac{1}{2} T u_x^2 \) for the elastic energy density. We do this by calculating how much energy it takes to deform the string from its straight equilibrium shape, to some shape \( y = u(x) \) (there is no time in this calculation). Imagine doing this by going left to right as follows: At any stage we assume that we have deformed the string so that it has the shape \( y = u(x) \) for \( x \leq \chi \), and \( y = u(\chi) \) for \( x \geq \chi \). Then we lift (up or down, but keeping it straight) the string in the region \( x \geq \chi + dx \), keeping the region \( x \leq \chi \) untouched. We do this till the differential segment \( [\chi, \chi + dx] \) achieves the slope \( u_x(\chi) \), and the string shape becomes \( y = u(x) \) for \( x \leq \chi + dx \), and \( y = u(\chi + dx) \) for \( x \geq \chi + dx \). The work done on the string at each stage is, precisely,

\[
\frac{1}{2} T u_x^2(\chi) dx = \int_0^{u_x(\chi)} dx \frac{T s}{dx} ds.
\]

This because the process involves applying the force \( T \times \) string slope = \( T s/dx \), at \( x = \chi + dx \), along the path \( y = u(\chi) + s \), \( 0 \leq s \leq u_x(\chi) dx \). Note that, in agreement with 3 in (8.1), we neglect any forces that the introduction of the kinks at \( x = \chi \) and \( x = \chi + dx \) may produce.

8.3.2 Impedance matching (extra tasks)

Imagine a semi-infinite string, \( 0 \leq x < \infty \), with a mass attached to the \( x = 0 \) end — as in item (vi) earlier. However, assume that the mass is tied up by a spring to \( x = y = 0 \). This yields an additional force, \( f_s = -k_s u(0, t) \), on the mass — here \( 0 < k_s = \text{spring constant} \). The boundary condition (8.6) must be modified to

\[
M u_t(0, t) + b f u_t(0, t) + k_s u(0, t) = T u_x(0, t).
\]

(8.10)

viii) Task: write an equation for the evolution of the total energy in the system, assuming that \( u_x \) and \( u_t \) vanish rapidly as \( x \to \infty \) (this guarantees that the total energy is finite).

Hint. Equation (8.9) should be useful. Note that the mass at the origin carries energy as well.
Consider now a wave of frequency \( \omega \), originating far away to the right, and traveling towards the origin. This wave, when it reaches the origin, excites vibrations on the mass-spring-dashpot system there, as well as a reflected wave traveling to the right. The solution corresponding to this scenario has the form \( u = \exp \left( i \omega \left( t + \frac{x}{c} \right) \right) + R \exp \left( i \omega \left( t - \frac{x}{c} \right) \right) \), (8.11)

where \( R \) = reflection coefficient.

\textbf{ix) Task: calculate} \( R \), \textbf{and show that} \( 0 \leq |R|^2 < 1 \).

\textit{Hint.} Since (8.11) solves (8.3), you only need to plug-in (8.11) into (8.10) to find \( R \).

\textbf{x) Task: show that} \( T \) \textbf{and} \( M \) \textbf{can be selected so that there is no reflected wave.} In this case the mass absorbs all the energy from the incident wave, and dissipates it via the frictional forces. \textbf{Express the required values for} \( T \) \textbf{and} \( M \) \textbf{in terms of} \( k_s \), \( \omega \), \( b_f \) \textbf{and} \( \rho \).

8.4 Statement: \textit{sv2ds02} (vibrations in a viscous fluid)

Here we consider the simple situation, where (on the string) there is no other force beside the tension \( T \), nor any dissipation. However, the string is vibrating in a viscous fluid, which produces a force (per unit length) opposing to the direction of motion. Assume that this force per unit length has the simple form \( -\nu_f u_t \), (8.12)

where \( \nu_f > 0 \) is a constant and \( u_t \) is the string velocity. Under these conditions, plus those in \( \S \) 8.1:

\textbf{Task #1. Use the conservation of the transversal momentum to derive an equation for} \( u \). Thus:

i) \textbf{Obtain a formula for the transversal momentum density along the string (momentum per unit length).}

ii) \textbf{Obtain a formula for the transversal momentum flux along the string (momentum per unit time).}

iii) \textbf{Use the differential form of the conservation of transversal momentum to write a pde for} \( u \).

\textit{Note that there is a momentum source as well.}

\textit{Hint. Remember that internal string forces} \( ^{11} \) \textit{are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces. Be careful with the sign here! Remember also that} \( u_x \) \textit{is very small.}

\textbf{Task #2. Use the conservation of energy to derive another equation for} \( u \). Furthermore, show that the solutions to the equation derived in task #1 satisfy the conservation of energy equation.

\textit{Hint. The energy density is the sum of the kinetic energy per unit length} \( (e_k) \), \textit{and the elastic energy per unit length stored in the string deformation} \( (e_d) \). \textit{You will need an expression for} \( e_d \). \textit{In fact, you only need to compute the difference between the elastic energy, and some constant energy — specifically: the elastic energy of the string at equilibrium. But the tension is constant (see} \( \S \) 8.1) \textit{over the stretching process taking the string from equilibrium} \( y = 0 \) \textit{to} \( y = u(x, t) \). \textit{Thus the elastic energy stored is the product of} \( T \) \textit{times the change in length of each segment} \( dx \). \textit{All you need now is the formula for the change in length of any interval} \( dx \) \textit{(actually, just the leading order approximation to this, in the} \( u_x \) \textit{small limit).}

\textit{This yields} \( e_d = T \left( \sqrt{1 + u_x^2} - 1 \right) = \frac{1}{2} T u_x^2 \), \textit{since the arc-length is} \( ds = \sqrt{1 + u_x^2} \, dx \), \textit{and} \( u_x \) \textit{is small.}

\textbf{Task #3. Assuming that the string is tied at both ends} \( u(0, t) = u(L, t) = 0 \), \textbf{write an equation for the evolution of the total energy}. \textit{That is:} \( \frac{dE}{dt} = ? ? \)

8.5 \textbf{Answer: sv2ds02}

We have

\textit{i) The transversal momentum per unit length along the string is given by} \( \rho_{\text{mom}} = \rho u_t \).

\( ^{10} \) Actually, the real part of this.

\( ^{11} \) Internal forces = forces by one part of the string on another.
ii) **Momentum flux along the string** happens because of transversal forces. That is: the force in the $y$-direction that, at every point, one side of the string applies on the other side. This is just the $y$-component of the tension, which is given by $T \sin \theta$, where $\theta$ is the angle that the string tangent makes with the $x$-axis. However, since $u_x$ is small, $\theta \approx u_x$. Thus the transversal momentum flux is given by

$$q_{\text{mom}} = -T u_x.$$  

Note about the sign here: when $u_x < 0$, the left side of the string pulls the right side up, generating a positive momentum flux. Vice-versa, when $u_x > 0$, momentum flows from right to left (negative). Thus the sign is as above.

Note also that the $x$-component of the tension (horizontal force along the string) is $T \cos \theta = T$, since $u_x$ is small. Thus it is constant, consistent with the approximation of no motion in the $x$ direction.

In addition, there is a momentum sink, provided by the resistance that the liquid opposes to the motion of the string. This is given by (8.12).

iii) **Equation for the conservation of the transversal momentum**

It must be $(\rho_{\text{mom}})_t + (q_{\text{mom}})_x = \text{sources/sinks}$. Thus

$$\rho \frac{\partial u_t}{\partial t} - T \frac{\partial u_x}{\partial x} = -\nu_f u_t.$$  

Alternatively

$$\frac{\partial u_t}{\partial t} - c^2 \frac{\partial u_x}{\partial x} = -\mu_f u_t,$$  

where $c^2 = T/\rho$ is a speed (the wave speed) and $\mu_f = \nu_f/\rho$.

Note that $1/\mu_f$ is a time, a decay time scale.

iv) **Equation for the conservation of energy**

Using the hint for task #2 we see that

$$e_d = T \left( \sqrt{1 + u_x^2} - 1 \right) = \frac{1}{2} T u_x^2,$$

$$e_k = \frac{1}{2} \rho u_t^2,$$

$$q_e = -T u_x u_t.$$

Finally, we need to consider the work done by the fluid as follows from (8.12). This term is an energy sink. Thus the equation for the conservation of energy is

$$\left( \frac{1}{2} \rho u_t^2 + \frac{1}{2} T u_x^2 \right)_t - (T u_x u_t)_x = -\nu_f u_t^2.$$  

It is easy to see that (8.13) guarantees this.

*Why is energy conservation “automatic” for this model?* The reason is that, in this model, there is no mechanism for energy exchanges between mechanical and internal in the string, thus there is no internal energy contribution to the motion. Note, however, that (in the fluid) mechanical energy is being converted into heat. The fluid then acts as an external energy sink for the string.

v) **Time evolution of the total energy for a string tied at both ends, with $u = 0$ there.**

The total energy is $E = \int_0^L \left( \frac{1}{2} \rho u_t^2 + \frac{1}{2} T u_x^2 \right) dx$, and

$$\frac{dE}{dt} = -\nu_f \int_0^L u_t^2 dx \leq 0,$$  

as follows from (8.15). Note that this formula remains valid if $u = 0$ is replaced at either (or both) ends by $u_x = 0$.

**Remark 8.2** In item (v) above, imagine that the left end of the string is free to slide along a rod, but that there is friction opposing the sliding. Then the boundary condition $b_f u_t(0, t) = T u_x(0, t)$ applies, where $b_f > 0$ is the friction coefficient. In this case

$$\frac{dE}{dt} = -b_f u_t^2(0, t) - \nu_f \int_0^L u_t^2 dx \leq 0.$$  

(8.17)
which encodes the extra dissipation produced by friction along the rod. The extra term arises from the energy flux at the origin, given by $-T u_x u_t = -b_f u_t^2$. A similar equation arises if the right end of the string is allowed to slide over a rod.

8.5.1 Conservation of other quantities

In the derivation above we obtained an equation for the string using, solely, the conservation of the transversal momentum. We already saw that then energy is conserved as well (provided that we account for the energy lost to the fluid bath). There are other quantities that should be conserved as well. Let us check that this is so.

**Conservation of mass.** This is guaranteed by the parameterization we use. That is, each point $(x, u)$ follows a specific bit of the spring, with mass $\rho dx$. Mass is automatically conserved. This is typical of solid mechanics problems, where the parameterization tracks points in the object. In fluids individual particles are not tracked; instead the flow velocity at each point in space is prescribed. Thus mass conservation is not automatic, and must be enforced. On the other hand, in fluids the equations involve only first time derivatives of the densities — while in solids second order equations occur. This is because the velocities are variables in fluid problems, while in solids they result from the time derivatives of the displacements.

**Conservation of longitudinal momentum.** Since there is no motion in the $x$-direction, and the horizontal component of the tension forces is constant (see the end of item (ii) of the answer), the longitudinal momentum is conserved as well (trivially so). In fact, _the statement that the derivation above does not use the conservation of the longitudinal momentum is not quite correct!_ This is used, effectively, when concluding that the tension on the string is constant.

8.5.2 Simple uniqueness proof for the initial value problem

Let $u_1$ and $u_2$ be two solutions of (8.13) with the same initial values, and the same Dirichlet/Neumann boundary conditions: (L) $u_1(0, t) = u_2(0, t)$ or $(u_1)_x(0, t) = (u_2)_x(0, t)$; and (R) $u_1(L, t) = u_2(L, t)$ or $(u_1)_x(L, t) = (u_2)_x(L, t)$.

Assume also that $u_1$ and $u_2$ are twice continuously differentiable (this for simplicity). Then $u_1 = u_2$.

The proof is very simple. Because the problem is linear, $u = u_1 - u_2$ satisfies the homogeneous problem, thus (8.16) applies. But then, since $\mathcal{E}(0) = 0$, and $\mathcal{E} \geq 0$, it follows from (8.16) that $\mathcal{E} \equiv 0$. Hence $u_t \equiv 0$. But $u = 0$ initially. Thus $u \equiv 0$. ♣

THE END.