Various lecture notes for 18306.

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April 14, 2014

Abstract

Notes for MIT's 18.306 Advanced PDE with applications.

Contents

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1 First lecture.

Extracts from topics covered:

(01) **Tomography.** Describe context and radon transform. Problem not well posed if point values required. Appropriate filtering (convolution with a kernel of the answer) gives a good problem.

(02) Initial value problem for ode. Quote theorem: existence, uniqueness, and continuous dependence on the initial value and equation parameters. Nothing at this level of generality exists for pde.

(03) Boundary value problems for ode. Situation more complicated. No general theorems exist at the level of item 02. Simple example: $\ddot{y} + y = 0$, for $0 < t < \ell$, with boundary conditions $y(0) = y(\ell) = 0$. Then there is no solution unless $\ell = n\,\pi$ with $n \in \mathbb{N}$. In this later case, there are infinitely many solutions $y = a\,\sin(t)$ — a an arbitrary constant.

(04) A pde version of the example in item 03 is the Poisson equation with Neumann B.C. Let Ω be a region[1] in $\mathbb{R}^d$, with boundary $\partial\Omega$, and unit outside normal $\hat{n}$. Consider now the problem

$$\Delta\, u = f(\vec{x}) \quad \text{for} \;\; \vec{x} \in \Omega, \tag{1.1}$$

with boundary condition $\hat{n} \cdot \nabla u = g(\vec{x})$ for $\vec{x} \in \partial\Omega$, where f and g are some given functions. If there is a solution, then

$$\int_\Omega f\ \mathrm{d}V = \int_\Omega \Delta u\ \mathrm{d}V = \int_{\partial\omega} \hat{n} \cdot \nabla u\ \mathrm{d}S = \int_{\partial\omega} g\ \mathrm{d}S, \tag{1.2}$$

where $\mathrm{d}V$ and $\mathrm{d}S$ denote the elements of "volume" and "area", respectively, and we have used Gauss theorem for the middle equality in (1.2). Hence: for this problem to have a solution, the data must satisfy a solvability condition.

(05) Define pde. Linear and nonlinear.
Scalar, 1-space & 1-time, 1^{st} order. From general to simple:

– $F(u_t,\, u_x,\, u,\, x,\, t) = 0$.	Most general pde.
– $u_t = F(u_x,\, u,\, x,\, t)$.	Most general explicit evolution pde.
– $u_t = F(u,\, x,\, t)\, u_x + G(u,\, x,\, t)$.	Quasilinear (explicit evolution) pde.
– $u_t = F(x,\, t)\, u_x + G(u,\, x,\, t)$.	Semilinear (explicit evolution) pde.
– $u_t = F(x,\, t)\, u_x + G(x,\, t)\, u + H(x,\, t)$.	Forced linear (explicit evolution) pde.
– $u_t = F(x,\, t)\, u_x + G(x,\, t)\, u$.	Homogeneous linear (explicit evolution) pde.

Further simplification occur if the equations are independent of x and t *(constant coefficients).*

[1] Assume that the region is as nice as needed, e.g.: the image of a ball by a diffeomorphism.

(06) The solutions to linear, constant coefficients, equations can be written as a superposition of elementary solutions of the form[2] $\boldsymbol{u} = \boldsymbol{A}\, \boldsymbol{e}^{\boldsymbol{i\,(kx-\omega\, t)}}$, where A is a complex constant, k is a real number, and ω must satisfy an equation of the form $G(k,\, \omega) = 0$, for some function G — note that there can be more than one branch of solutions for ω.

Assume now that ω is real valued. Then the equation is a *wave system* and

- k is the *wave number* and $\lambda = \frac{2\pi}{k}$ is the *wave length.*
- ω is the *wave frequency* and $T = \frac{2\pi}{\omega}$ is the *wave period.* The frequency is $f = \frac{1}{T} = \frac{\omega}{2\pi}$.
- $a = |A|$ is the *wave amplitude.*
- $\theta = kx - \omega t + \phi$ is the phase, where ϕ is the polar angle for A.
- If $\frac{d^2\omega}{dk^2} \neq 0$, the system is called dispersive, and G is the *dispersion relation.*

Example: $u_t + u_x - u_{xxx} = 0$ yields $\omega = k + k^3$.
Example: $u_{tt} - u_{xx} + u = 0$ yields $\omega^2 = 1 + k^2$.

(07) Derivation of pde using **Conservation Laws.** The continuum limit. Closure problem: quasi-equilibrium and thermodynamics. Integral and differential forms of a conservation law.

(08) Start with **scalar 1^{st}-order quasi-linear pde and the theory of characteristics.**

2 Laplace and Poisson equations - harmonic functions.

This section contains notes regarding harmonic functions and the Laplace (and Poisson) equations.

2.1 PPE formulations for the Navier Stokes equations. Not yet done.

PPE reformulations of the Navier-Stokes equations, and the boundary conditions that they produce for the Poisson equation that the pressure satisfies.

2.2 Mean value theorem, etc.

2.2.1 Poisson equation. Uniqueness.

We consider here the Poisson equation

$$\Delta u = f \tag{2.2.1}$$

in a sufficiently nice,[3] connected and bounded, region Ω — with a boundary $\partial\Omega$ and outside unit normal $\hat{n}$. Assume that Dirichlet, Neumann, or Robin boundary conditions apply (or perhaps a

[2] For simplicity, we consider here a scalar, one dimensional, problem.
[3] Basically: nice enough to be able to use Gauss theorem.

combination of these, each valid on a different part of the boundary). Finally, we will restrict ourselves to solutions that are twice continuously differentiable inside Ω, with the function and its derivatives continuous up to the boundary.

Under these conditions, the solutions to (2.2.1) are unique, with the proviso that: in the case of pure Neumann b.c., uniqueness applies up to an additive constant. (2.2.2)

To show this, we being by noticing that: if u_1 and u_2 are two solutions, then $v = u_1 - u_2$ satisfies the Laplace equation in Ω, with homogeneous boundary conditions. Hence

$$0 = \int_\Omega v\,\Delta\,v\,dV = \int_{\partial\Omega} v\,v_n\,dA - \int_\Omega (\text{grad}\,v)^2\,dV \tag{2.2.3}$$

where $v_n = \hat{n}\cdot\text{grad}\,v$ is the outside normal derivative along the boundary. Then:

1. Pure Dirichlet boundary condition case. Then v vanishes on the boundary and (2.2.3) reduces to $\int_\Omega(\text{grad}\,v)^2\,dV = 0 \implies \text{grad}\,v = 0$ in $\Omega \implies v = \text{constant}$ in Ω. However, v vanishes in $\partial\Omega$. Hence v vanishes everywhere. ♣

2. Pure Neumann boundary condition case. Then v_n vanishes on the boundary, and the same argument as in case **1** applies, except that we can only conclude that $v = \text{constant}$. ♣

3. Pure Robin boundary condition case. Then $v_n + \alpha\,v = 0$ on the boundary, for some $\alpha \geq 0$ (note that α could be a function,[4] it need not be a constant). Substituting $v_n = -\alpha\,v$ in (2.2.3) we obtain $\int_\Omega(\text{grad}\,v)^2\,dV + \int_{\partial\Omega}\alpha\,v^2\,dA = 0$, from which we conclude that $\text{grad}\,v = 0$ in Ω, and $\alpha\,v^2 = 0$ in $\partial\Omega$. The rest is now as in case **1**. ♣

4. Mixed boundary condition case. Follows by putting together all the arguments above. ♣

Example. What happens if the condition $\alpha \geq 0$ does not hold in case **3**?
Consider the 1-D version of the problem: $v'' = 0$ for $0 < x < 1$, with $v'(0) - \alpha_1\,v(0) = 0$ and $v'(1) + \alpha_2\,v(1) = 0$ — where $\alpha_j \geq 0$, and at least one of them is not zero [#]. The general solution to this problem is $v = a + b\,x$, with $-\alpha_1\,a + b = 0$ and $\alpha_2\,a + (1+\alpha_2)\,b = 0$. Uniqueness occurs if and only if $0 \neq \alpha_1\,(1+\alpha_2) + \alpha_2$. This is guaranteed by [#]. However, if this is not required, $\alpha_1\,(1+\alpha_2) + \alpha_2 = 0$ is possible — e.g.: $(\alpha_1,\,\alpha_2) = (-0.5,\,1)$, $(\alpha_1,\,\alpha_2) = (2,\,-2)$, and $(\alpha_1,\,\alpha_2) = (-3,\,-1.5)$. That is: the solutions may be unique, but this is not guaranteed.
One way to understand the situation is to notice that, if $\alpha < 0$ somewhere, then the operator $\mathcal{L} = -\Delta$, while still self-adjoint, is no longer definite and has negative, and positive, eigenvalues — in particular, it may have zero as an eigenvalue.

[4] If $\alpha \equiv 0$ then this case reduces to case **2**. We **exclude** this possibility here.

2.2.2 Intuition for the mean value theorem.

Consider a square grid in some region within $\mathcal{R}^2$, $x_n = x_0 + n\,h$ and $y_m = x_0 + m\,h$, with n and m integers. Then a second order approximation to the Laplacian operator is given by the 5-point stencil formula

$$(\mathcal{L}\,u)_{n\,m} = \frac{1}{h^2}\,(u_{n+1\,m} + u_{n-1\,m} + u_{n\,m+1} + u_{n\,m-1} - 4\,u_{n\,m}) \tag{2.2.4}$$

where $u_{n\,m}$ denotes the value at $(x_n\;y_m)$ of the discrete function u. A *discrete harmonic function* is a discrete function that satisfies $\mathcal{L}\,u = 0$. It is defined by the property that: *the value of u at a grid point is the average of the values at the neighboring points.* From this it also follows that: *the maximum and minimum values that u achieves in any region occur at the boundary,* and that: *if the maximum (minimum) of u occurs at an interior point of a connected region, then u is constant in the region.* These properties carry over to continuum harmonic functions, as we will show next.

Note #1. Formulas analogous to (2.2.4) apply in all dimensions.

Note #2. The book by Salsa (see course syllabus) develops the analogy between the discrete and the continuous Laplacian in some detail. See § 3.3.1, pp. 105–109.

2.2.3 The mean value theorem in 2-D.

Here we consider the mean value theorem, and its consequences, for harmonic functions in 2-D. Extensions to n dimensions are straightforward.

We say that **a function is harmonic** in some region if *it is twice continuously differentiable, and it satisfies the Laplace equation.*

Let $\boldsymbol{u}$ **be an harmonic function in some open set** $\boldsymbol{\Omega}$ and take $P_0 = (x_0,\, y_0) \in \Omega$. Then, for any sufficiently small radius $r \geq 0$ we can define the mean value of u over the circle of radius r centered at P_0

$$M(u,\, x_0,\, y_0,\, r) = \frac{1}{2\,\pi}\int_0^{2\,\pi} u(x_0 + r\,\cos\theta,\, y_0 + r\,\sin\theta)\,d\theta \tag{2.2.5}$$

Now we have

$$\begin{aligned} \frac{\partial M}{\partial r} &= \frac{1}{2\,\pi}\int_0^{2\,\pi}(\cos\theta\,u_x + \sin\theta\,u_y)\,d\theta \;=\; \frac{1}{2\,\pi\,r}\int_{C(r,\,x_0,\,y_0)} u_n\,ds \\ &= \frac{1}{2\,\pi\,r}\int_{D(r,\,x_0,\,y_0)} \mathrm{div}(\mathrm{grad}u)\,dx\,dy \;=\; 0, \end{aligned} \tag{2.2.6}$$

where: (i) $\boldsymbol{C(r,\,x_0,\,y_0)}$ is the circle of radius r centered at P_0. (ii) $\boldsymbol{u_n}$ denotes the normal outer derivative of u on $C(r,\, x_0,\, y_0)$. (iii) $\boldsymbol{s}$ is the arc-length on $C(r,\, x_0,\, y_0)$. (iv) $\boldsymbol{D(r,\,x_0,\,y_0)}$ is the disk of radius r centered at P_0. Since $M_c(u,\, x_0,\, y_0,\, 0) = u(x_0,\, y_0)$, we conclude that

$$u(x_0,\, y_0) = \frac{1}{2\,\pi}\int_0^{2\,\pi} u(x_0 + r\,\cos\theta,\, y_0 + r\,\sin\theta)\,d\theta. \tag{2.2.7}$$

This is the **mean value theorem for harmonic functions in 2-D.**

Integrating (2.2.7) over r, it is easy to see that

$$u(x_0,\, y_0) = \frac{1}{\pi\, r^2} \int_0^r \rho\, d\rho \int_0^{2\,\pi} u(x_0 + \rho\, \cos\theta,\, y_0 + \rho\, \sin\theta)\, d\theta = \frac{1}{\pi\, r^2} \int_{D(r,\, x_0,\, y_0)} u\, dx\, dy, \tag{2.2.8}$$

which is an **alternative formulation for the mean value theorem.**

Remark 2.2.1 *The* **two formulations** *of the mean value theorem are* **equivalent:** *a function u satisfies (2.2.7) for all (small enough) r if, and only if, it satisfies (2.2.8) for all (small enough) r.*

Proof. We have already show that (2.2.7) implies (2.2.8). To see the reverse, multiply the two left-most terms in (2.2.8) by r^2, differentiate with respect to r, and divide by r. ♣

Corollary of the mean value theorem: the maximum/minimum principle.

> *An harmonic function achieves its maximum and minimum values over the boundary of any region over which it is defined. If the maximum (or minimum) occurs at an interior point, then the function is a constant.* (2.2.9)

Note that this principle can be used to provide a proof of uniqueness for the solution of a Poisson problem with Dirichlet boundary conditions.

Let now u **be a continuous function in some open set Ω, which satisfies the mean value theorem.** That is: for any point $P_0 = (x_0,\, y_0) \in \Omega$, and any $r > 0$ (small enough so that the disk of radius r centered at P_0 is included within Ω), equations (2.2.7–2.2.8) apply. We show next that:

$$\left.\begin{array}{ll} (a) & \textit{The function } u \textit{ has infinitely many (continuous) partial derivatives.} \\ (b) & \textit{Every partial derivative of } u \textit{ satisfies the mean value property.} \\ (c) & \textit{The function } u \textit{ is harmonic.} \end{array}\right\} \tag{2.2.10}$$

Since harmonic functions satisfy the mean value theorem, it follows that

Harmonic functions have infinitely many partial derivatives, each itself harmonic. (2.2.11)

This is a surprising result, given that to be harmonic satisfying a pde involving second derivatives only is required.

Proof (2.2.10). Take a point $P_0 = (x_0,\, y_0) \in \Omega$, and select $R > 0$ small enough so that the disk of radius $2\,R$ centered at P_0 is included in Ω. Then, for any h small enough, using (2.2.8) we can write

$$u(x_0 + h,\, y_0) = \frac{1}{\pi\, R^2} \int_0^{2\,\pi} d\theta \int_0^r \rho\, d\rho\, u(x_0 + \rho\, \cos\theta,\, y_0 + \rho\, \sin\theta), \tag{2.2.12}$$

where $r = r(h,\,\theta)$ parameterizes the circle of radius R centered at $(x_0+h,\,y_0)$, in terms of the polar coordinates centered at P_0. That is:

$$r = h\,\cos\theta + \sqrt{R^2 - h^2\,(1-\cos^2\theta)} = R + h\,\cos\theta + O(h^2). \tag{2.2.13}$$

Thus, differentiating (2.2.12) with respect to h we obtain

$$u_x(x_0,\,y_0) = \frac{1}{\pi\,R}\int_0^{2\,\pi} u(x_0 + R\,\cos\theta,\,y_0 + R\,\sin\theta)\,\cos\theta\,d\theta. \tag{2.2.14}$$

This shows that u_x exists, and that it is continuous. Similarly, it can be shown that

$$u_y(x_0,\,y_0) = \frac{1}{\pi\,R}\int_0^{2\,\pi} u(x_0 + R\,\cos\theta,\,y_0 + R\,\sin\theta)\,\sin\theta\,d\theta. \tag{2.2.15}$$

That is: u is continuously differentiable at least once. Knowing this, we use (2.2.7) to write

$$u_x(x_0,\,y_0) = \frac{1}{2\,\pi}\int_0^{2\,\pi} u_x(x_0 + r\,\cos\theta,\,y_0 + r\,\sin\theta)\,d\theta, \tag{2.2.16}$$

and similarly for u_y. Thus the first partial derivatives of u exists, are continuous, and satisfy the mean value property. Iterating the procedure above, (a) and (b) in (2.2.10) follow.

Finally, differentiate (2.2.7) with respect to r. This yields, upon use of Gauss theorem,

$$0 = \int_0^{2\,\pi} (\cos\theta\,u_x + \sin\theta\,u_y)(x_0 + r\,\cos\theta,\,y_0 + r\,\sin\theta)\,r\,d\theta = \int_{D(r,\,x_0,\,y_0)} \Delta\,u\,dx\,dy. \tag{2.2.17}$$

However Δu satisfies the mean value property (since both $u_{x\,x}$ and $u_{y\,y}$ do). Thus $\Delta u(x_0,\,y_0) = 0$, which proves (c) in (2.2.10). ♣

It can be shown that *equations similar to (2.2.14) and (2.2.15) apply to all the partial derivatives of u.* That is, all of them can be written as integrals of u (with an appropriate weight) over circles.

2.3 Poisson's formula and Harnack's inequality.

For simplicity we consider here only harmonic functions [5] in 2-D, but everything here has a straightforward extension to higher dimensions. *Let u be harmonic inside a disk of radius $R > 0$, continuous up to the boundary.* Without loss of generality, assume that the disk is given by $r < R$, where r is the polar radius in the plane. Then

$$u(r,\,\theta) = \int_0^{2\,\pi} G(r,\,\theta-\phi)\,u(R,\,\phi)\,R\,d\phi, \tag{2.3.1}$$

where θ is the polar angle and

$$G(r,\,\theta) = \frac{1}{2\,\pi\,R}\,\frac{R^2 - r^2}{R^2 - 2\,R\,r\,\cos(\theta) + r^2} \tag{2.3.2}$$

[5] Recall that harmonic means: twice continuously differentiable satisfying the Laplace equation.

is the *Poisson* kernel [6] — in particular: *u is C^∞ inside the disk.* In cartesian coordinates this takes the form

$$u(\vec{x}) = \int_{|\vec{y}|=R} G(\vec{x},\, \vec{y})\, u(\vec{y})\, ds(\vec{y}), \quad \text{where} \quad G = \frac{1}{2\,\pi\,|\vec{y}|}\,\frac{|\vec{y}|^2 - |\vec{x}|^2}{|\vec{x}-\vec{y}|^2} \tag{2.3.3}$$

and s is the arc-length along the circle of radius R. Poisson's formula has the following consequences

1. **Harnack's inequality.** Let u be harmonic and *non-negative* for $r < R$, continuous for $r \le R$. Then $$\boldsymbol{\frac{R-r}{R+r}\, u(0) \le u(\vec{x}) \le \frac{R+r}{R-r}\, u(0)} \quad \text{for any } \vec{x} \text{ such that } |\vec{x}| = r < R. \tag{2.3.4}$$

Proof. From (2.3.2) $G \le \dfrac{1}{2\,\pi\,R}\,\dfrac{R^2-r^2}{(R-r)^2} = \dfrac{1}{2\,\pi\,R}\,\dfrac{R+r}{R-r}$ and $G \ge \dfrac{1}{2\,\pi\,R}\,\dfrac{R^2-r^2}{(R+r)^2} = \dfrac{1}{2\,\pi\,R}\,\dfrac{R-r}{R+r}$.

Then $u \ge 0$ and (2.3.1) give: $$\frac{1}{2\,\pi}\,\frac{R-r}{R+r}\int_0^{2\,\pi} u(R,\,\phi)\,d\phi \le u(\vec{x}) \le \frac{1}{2\,\pi}\,\frac{R+r}{R-r}\int_0^{2\,\pi} u(R,\,\phi)\,d\phi.$$

Equation (2.3.4) follows now from the mean value theorem. ♣

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2. **Liouville's theorem.** If u is harmonic and bounded (from either above, or below) in $\mathcal{R}^2$, then *u is a constant.*

Proof. Assume that $u \ge M$, for some constant M. Then (2.3.4) applies to $v = u - M$ for any $R > 0$. The limit $R \to \infty$ yields $v(0) \le v(\vec{x}) \le v(0)$, i.e.: $v \equiv v(0)$. For $u \le M$ use $v = M - u$. ♣

Finally, note that (2.3.1) makes sense even if $h(\phi) = u(R,\,\phi)$ is not continuous, but merely square integrable. In this case $u \to h$ as $r \to R$ in the L^2 sense only (not point-wise).

2.4 The fundamental solutions.

It is easy to check that the rotationally invariant solutions to the Laplace equation have the form

$$u = c_1\,\ln(r) + c_2 \quad \text{in 2-D, and} \quad u = c_1\, r^{2-n} + c_2 \quad \text{in n-D, } n > 2, \tag{2.4.1}$$

where r is the radial variable and c_1 and c_2 are constants. In particular

$$\Phi = -\frac{1}{2\,\pi}\,\ln(r) \quad \text{in 2-D, and} \quad \Phi = \frac{1}{4\,\pi\,r} \quad \text{in 3-D,} \tag{2.4.2}$$

satisfy

$$\Delta\,\Phi = -\delta(\vec{x}), \tag{2.4.3}$$

[6] For $R = 1$ this formula is proved in the *Laplace equation in a circle #01* problem. The general case follows by scaling, since being harmonic is invariant under radial stretching in the plane — i.e.: define $v(\vec{x}) = u(\vec{x}\,R)$, $|\vec{x}| \le 1$.

and are called **the fundamental solutions.**[7]

Proof in 2-D. We need to show that, for any test function ψ, $\displaystyle\int \Phi\,\Delta\psi\,dx_1\,dx_2 = -\psi(\vec{0})$. To show this, define $\overline{\psi} = \frac{1}{2\pi}\int_0^{2\pi}\psi(r,\,\theta)\,d\theta$. Then $\displaystyle I = \int \Phi\,\Delta\psi\,dx_1\,dx_2 = -\frac{1}{2\pi}\int_0^\infty r\,\ln(r)\,dr\,\underbrace{\int_0^{2\pi}\left(\frac{1}{r}\,(r\,\psi_r)_r + \frac{1}{r^2}\,\psi_{\theta\,\theta}\right)d\theta}_{A}$.

However $A = \dfrac{2\pi}{r}\,(r\,\overline{\psi}_r)_r$. Thus $\displaystyle I = -\int_0^\infty \ln(r)\,(r\,\overline{\psi}_r)_r\,dr = \int_0^\infty \overline{\psi}_r\,dr = -\overline{\psi}(0) = -\psi(\vec{0})$. ♣

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3 Scalar first order quasilinear pde and characteristics.

In this section we consider the class of pde of the general form

$$\vec{a}\cdot\nabla u = \sum_j a_j(u,\,\vec{x})\,u_{x_j} = b(u,\,\vec{x}), \tag{3.1}$$

where $\vec{x}\in\mathbb{R}^n$, the unknown $u = u(\vec{x})$ is a scalar valued function, and $\vec{a}$ and b are given functions.

3.1 Examples.

Example 3.1 Consider the equation $\boldsymbol{u_t + c\,u_x = 0}$, where $\boldsymbol{c \neq 0}$ is a constant.
The equation states that a certain directional derivative (in space-time) of the solution vanishes. Specifically:

$$\text{Along the curves }\ \frac{dx}{dt} = c,\quad \text{the solution is constant }\ \frac{du}{dt} = 0. \tag{3.2}$$

Solving the (trivial) system of two ode in (3.2) yields $x = x_0 + c\,t$ and $u = U_1(x_0)$, where x_0 is a label for the curves. We can also write $x = c\,(t - t_0)$ and $u = U_2(t_0)$ — using a different label, related to the prior one by $x_0 = -c\,t_0$. From this we can see that the general solution to the equation in (3.2) can be written in either of the following two forms

$$u = U_1(x - c\,t) \quad\text{or}\quad u = U_2\left(t - \frac{x}{c}\right). \tag{3.3}$$

The first form is useful for initial value problems and the second for boundary value problems (signaling).

- *Discuss solution of the initial value problem $u(x,\,0) = U_0(x)$.*
- *Discuss solution of the boundary value problem $u(0,\,t) = \sigma(t)$.*
  Causality: on which side of the time axis does this determine the solution?

[7] The fundamental solutions for arbitrary dimensions are the subject of the problem *Fundamental solution for the Laplace operator* in the 306 problem series *Point Sources and Green functions.*

– *Discuss solution in a bounded domain: $0 < x < L$ and $t > 0$.* Where must the data be given? *Assume $c = 1$ and do the cases with: Dirichlet, Neumann, and periodic boundary conditions.* ***Draw the characteristics in space-time.***

Example 3.2 Consider the equation $\boldsymbol{u_t + c\, u_x = a\, u + b}$, with constants $\boldsymbol{c \neq 0}$, $\boldsymbol{a \neq 0}$, and $\boldsymbol{b}$. *As in example (3.1), the equation can be reduced to a system of ode along curves in space-time (the characteristics), Namely:*

$$\text{along } \frac{dx}{dt} = c, \quad \text{the solution satisfies } \frac{du}{dt} = a\,u + b. \tag{3.4}$$

Thus either .... $u = -\dfrac{b}{a} + \left(U_1(x_0) + \dfrac{b}{a}\right) e^{a\,t}$ *along* $x = c\,t + x_0$,

or .... $u = -\dfrac{b}{a} + \left(U_2(t_0) + \dfrac{b}{a}\right) e^{a\,(t-t_0)}$ *along* $x = c\,(t - t_0)$.

This yields

$$u = -\frac{b}{a} + \left(U_1(x - c\,t) + \frac{b}{a}\right) \exp\,(a\,t) \quad \text{or} \quad u = -\frac{b}{a} + \left(U_2\left(t - \frac{x}{c}\right) + \frac{b}{a}\right) \exp\left(\frac{a}{c}\,x\right). \tag{3.5}$$

Note: In situations like the two examples above (where t is time and x is space), the most natural set of data involves a mix of initial data and boundary data. This is, however, not the only possibility. In general, it should be clear that:

> **The data for the solution can be given along any curve in space, provided that the curve (transversaly) intersects every characteristic (in the region where the solution is desired) exactly once.** (3.6)

A clarification and an important proviso:

Important proviso: when time is involved, (3.6) still applies, as long as *causality is not violated.* Data along a curve cannot be used to determine the past. In this case the characteristics start on the curve with the data, and extend only forwards in time.

Clarification: In (3.6) **transversally** means that *the characteristics are not tangent (at the point of intersection) to the curve where the data is given.* To see why this is important, consider the equation in example 3.1, and assume that the solution is prescribed along some curve $x = \chi(t)$, as follows: $u = \sigma(t)$ for $x = \chi(t)$. Hence $u(\chi(t),\, t) = \sigma(t)$, which implies $u_t + \dot{\chi}\, u_x = \dot{\sigma}$ for $x = \chi(t)$. However, if $\dot{\chi}(t_0) = c$ at any point, then (from the equation) it should also be that $\dot{\sigma}(t_0) = 0$. It should be clear that what happens in this example is generic: At points where the intersecting characteristic is tangential to the curve where the data is prescribed, restrictions on the data are needed.

Example 3.3 Consider the equation $\boldsymbol{x\, u_x + y\, u_y = 0}$, for $\boldsymbol{y > 1}$ and $\boldsymbol{-\infty < x < \infty}$, with $\boldsymbol{u = F(x)}$ along $\boldsymbol{y = 1}$.

As in example (3.1), the equation can be reduced to a system of ode along curves in the plane (the characteristics), Namely:

$$\text{along } \frac{d}{dt}(x,\,y) = (x,\,y), \quad \text{the solution satisfies } \frac{du}{dt} = 0. \tag{3.7}$$

Note that here t is a parameter along the characteristics (it is NOT time). Solve this system of ode, with the initial condition (i.e.: for $t = 0$) given by the data: $x = x_0$, $y = 1$, and $u = F(x_0)$. Thus

$$(x,\,y) = (x_0,\,1)\,e^t \text{ and } u = F(x_0) \quad \Longrightarrow \quad u = F\left(\frac{x}{y}\right). \tag{3.8}$$

The characteristics in this case are the rays through the origin. Hence, this example corresponds to the situation described in equation (3.6). *Notice that the solution is, in fact, defined for all of $y > 0$ by the given data. Things fail for $y \leq 0$, however, because all the characteristics converge into the origin.* For this equation one cannot determine the solution for $y \leq 0$ from data given on $y > 0$, no matter what the data is, or where exactly it is given.

Example 3.4 Consider the equation $\boldsymbol{x\,u_x + y\,u_y = 1 + y^2}$, for $\boldsymbol{y > 1}$ and $\boldsymbol{-\infty < x < \infty}$, with $\boldsymbol{u = F(x)}$ along $\boldsymbol{y = 1}$.
As in example (3.1), the equation can be reduced to a system of ode along curves in the plane (the characteristics), Namely:

$$\text{along } \frac{d}{dt}(x,\,y) = (x,\,y), \quad \text{the solution satisfies } \frac{du}{dt} = 1 + y^2. \tag{3.9}$$

Again, here t is a parameter along the characteristics (it is NOT time). Solve this system of ode, with the initial condition ($t = 0$) given by the data: $x = x_0$, $y = 1$, and $u = F(x_0)$. Thus

$$(x,\,y) = (x_0,\,1)\,e^t \text{ and } u = F(x_0) + t + \frac{1}{2}\left(e^{2\,t} - 1\right) \quad \Rightarrow \quad u = \ln(y) + \frac{1}{2}\left(y^2 - 1\right) + F\left(\frac{x}{y}\right). \tag{3.10}$$

Again, this example corresponds to the situation described in equation (3.6).

Example 3.5 Write the equation $\boldsymbol{x\,u_x + y\,u_y = f}$ in polar coordinates.
A simple calculation shows that the equation can be written in the form

$$u_r = \frac{1}{r}\,f. \tag{3.11}$$

This clearly illustrates the fact that the characteristics are the rays through the origin.

3.2 Theory.

3.2.1 General procedure for solving linear 1^{st} order equations. Not yet done.

In this subsection we consider the solution by characteristics of first order linear equations of the form

$$a(x,\,y)\,u_x + b(x,\,y)\,u_y = c(x,\,y)\,u + d(x,\,y). \tag{3.12}$$

3.2.2 Domain of dependence and domain of influence. Not yet done.

3.2.3 Domain of definition, given the data. Not yet done.

3.2.4 Extension to semi-linear and quasi-linear equations. Not yet done.

3.2.5 Examples: Traffic Flow and Flood Waves in Rivers. Not yet done.

3.2.6 General definition of characteristics. Relationship with signal propagation and "weak" singularities. Not yet done.

4 Hamilton Jacobi equation.

4.1 Characteristics for the general first order scalar pde in 2-D.

The most general scalar first order pde in 2-D can be written in the form

$$H(u,\, p,\, q, x,\, y) = 0, \quad \text{where } \; p = u_x, \;\; q = u_y, \tag{4.1}$$

and H is some given function. Here we will assume that $\boldsymbol{H}$ is twice continuously differentiable.

4.1.1 Characteristics and weak singularities.

Let us now look for the characteristics of equation (4.1), using the (somewhat informal) idea that the characteristics are curves across which the equation allows the "propagation" of weak singularities.[8] To be more precise, we ask the following question:

$$\textbf{(Q1)} \left\{ \begin{array}{l} \text{Given a solution to the equation, are weak singularities} \\ \text{allowed in the "infinitesimal" perturbations to the solution?} \end{array} \right. \tag{4.2}$$

Remark 4.1 *First: why this? The intuitive motivation is that we want to take the solution, modify it somewhere by adding a very small imperfection (the signal), and see if this signal "propagates". Second: why not look directly for singularities in the solutions to the equation? The reason is that the equation may admit solutions with singularities which are not associated with characteristics — e.g.: shocks, see examples 4.1 and 4.2. The intuitive idea is that the "size" of the singularity along a characteristic is not "restricted" — e.g.: equation (4.3) is linear, so any multiple of a solution is also a solution. On the other hand, along shocks the singularities are restricted — e.g. Rankine-Hugoniot jump conditions and entropy conditions.*

To answer the question in (4.2), we inspect the equation for the perturbation

$$H_u\, \delta u + H_p\, (\delta u)_x + H_q\, (\delta u)_y = 0, \tag{4.3}$$

[8] Loosely, what we mean by a weak singularity in the solution to a pde is: all the derivatives needed to give meaning to the pde are defined, but some derivative (possibly of some high order) has a simple jump discontinuity across the curve.

where δu is the perturbation, and H, H_p, and H_q are evaluated at the known solution. From this last equation, we see that:

$$\left.\begin{array}{l}\text{The answer to (4.2) is yes. The weak singularities}\\ \text{in the infinitesimal perturbations can occur along}\end{array}\right\} \quad \frac{dx}{dt} = H_p \quad \text{and} \quad \frac{dy}{dt} = H_q. \tag{4.4}$$

The curves defined by the system of ode in (4.4) are the characteristics for equation (4.1).

Remark 4.2 *Notice that for the system of ode in (4.4) to actually have solutions, some smoothness restriction on u are needed. For example: u continuously differentiable, with Lipschitz continuous derivatives is enough to guarantee existence and uniqueness of a characteristic curve through every point $(x,\, y)$. Of course, this breaks down when the solution has shocks — see examples 4.1 and 4.2. Generally, the characteristics end at a shock.*

The next question is:

$$\textbf{(Q2)} \left\{\begin{array}{l}\text{What kind of weak singularities can the solutions}\\ \text{to equation (4.1) have along the characteristics?}\end{array}\right. \tag{4.5}$$

Remark 4.3 *For the equation to have meaning as written, the solution has to be at least differentiable. Note also that a jump discontinuity in ∇u is compatible with a solution only if $H = 0$ on both sides of the discontinuity — which (generally) restricts the type of jumps allowed. Further, for such solutions the characteristic curves may not even be defined where the discontinuities occur — see remark 4.2. Hence, when considering the question in (4.5) below, we exclude solutions where the gradient of u is not continuous*[9] *— however: see example 4.2.*

Let now u be a solution of equation (4.1), which we assume is twice differentiable with (at least) piecewise continuous second derivatives. Further:

> Let $\{\eta = \eta(x,\, y),\, \xi = \xi(x,\, y)\}$ be the coordinate functions for a (local) curvilinear coordinate system such that the curves $\eta =$ constant are characteristics. W.L.O.G. assume also that $\nabla\eta$ and $\nabla\xi$ are orthogonal. (4.6)

Let us now write equation (4.1) in terms of the $\{\eta,\, \xi\}$ coordinates

$$H(u,\, \eta_x\, u_\eta + \xi_x\, u_\xi,\, \eta_y\, u_\eta + \xi_y\, u_\xi,\, x,\, y) = 0, \tag{4.7}$$

where $x = x(\eta,\, \xi)$, $y = y(\eta,\, \xi)$, and we write $(\eta_x,\, \eta_y,\, \xi_x,\, \xi_y)$ as functions of $(\eta,\, \xi)$. This equation involves the gradient of u only. In order to detect (possible) weak singularities in higher order derivatives, we differentiate this equation (with respect to η and ξ). This yields

$$\frac{\partial H}{\partial u}\, u_\eta + \frac{\partial H}{\partial u_\eta}\, u_{\eta\,\eta} + \frac{\partial H}{\partial u_\xi}\, u_{\xi\,\eta} + \frac{\partial H}{\partial \eta} = 0 \tag{4.8}$$

[9] Solutions where the gradient of u is discontinuous can exist (see example 4.1). However: in general the locus where the discontinuity occurs should be considered a shock, not a characteristic.

and
$$\frac{\partial H}{\partial u}\,u_\xi+\frac{\partial H}{\partial u_\eta}\,u_{\eta\,\xi}+\frac{\partial H}{\partial u_\xi}\,u_{\xi\,\xi}+\frac{\partial H}{\partial \xi}=0. \tag{4.9}$$
Note also that η satisfies
$$0=\eta_x\,H_p+\eta_y\,H_q=\frac{\partial H}{\partial u_\eta}. \tag{4.10}$$
where the first equality follows from (4.4) and (4.6), and the second from (4.7). From this we see that an "unrestricted" jump discontinuity in $u_{\eta\,\eta}$ is possible along the characteristics (the curves where (4.10) applies). Hence *the answer to the question in (4.5) is:*

> We expect the weak singularities in the solutions to (4.1) to occur in the second order derivatives. Namely: the second order derivative in the direction normal to the characteristics can have an unrestricted simple jump discontinuity. (4.11)

See example 4.1, equation (4.19).

4.1.2 Solution independent ode system for the characteristics.

For first order quasi-linear equations, the characteristics can be set-up as a system of ode which can be solved without knowing the solution — that is, the characteristics can be found directly from the boundary/initial data, without prior knowledge of the actual solution to the pde. *This is also true for the more general case that we are considering here.* We show next that solving the pde equation in (4.1) is formally equivalent to solving a system of ode — one solution per characteristic. In order to do this, we have to augment the two ode in (4.4) to include equations for u, p, and q, since these variables are needed to calculate H_p and H_q. This is easily done, as shown below.

Consider now a solution $\boldsymbol{u}$ to (4.1) which is twice continuously differentiable. Then:
$$\left.\begin{array}{lrcl}
(1) & \dot{x} & = & H_p, \\
(2) & \dot{y} & = & H_q, \\
(3) & \dot{u} & = & p\,H_p+q\,H_q, \\
(4) & \dot{p} & = & -H_x-p\,H_u, \\
(5) & \dot{q} & = & -H_y-q\,H_u,
\end{array}\right\} \tag{4.12}$$
where: (i) The dots denote derivation with respect to the parameter t. (ii) Equation (3) follows from the chain rule $\dot{u}=u_x\,\dot{x}+u_y\,\dot{y}=p\,\dot{x}+q\,\dot{y}$, and use of the first two equations. (iii) Equation (4) follows from the chain rule and the first two equations $\dot{p}=p_x\,\dot{x}+p_y\,\dot{y}=p_x\,H_p+p_y\,H_q$, combined with the result of taking a partial derivative with respect to x of equation (4.1) — which yields $0=H_u\,u_x+H_p\,p_x+H_q\,q_x+H_x=H_u\,p+H_p\,p_x+H_q\,p_y+H_x$, since $p=u_x$ and $q_x=(u_y)_x=(u_x)_y=p_y$. (iii) Equation (5) follows in a similar way.

Finally, **consider a C^2 one parameter solution to the equations in (4.12)** — namely: $x = x(t,\, s)$, $y = y(t,\, s)$, etc. — **such that $H = 0$ and**[10] $\boldsymbol{\partial_s u = p\,\partial_s x + q\,\partial_s y}$ **for $t = 0$.** If $(x,\, y) = (x(t, s),\, y(t, s))$ can be inverted to write $(t,\, s) = (t(x, y),\, s(x, y))$, so that $u = u(x,\, y)$ — then:

$$\boldsymbol{p = u_x,\ q = u_y,\ \text{and } u \text{ solves (4.1).}} \tag{4.13}$$

Proof: From (4.12) $\partial_t H = \dot{H} = 0$, hence $H = 0$ for all $(t,\, s)$. Furthermore, some simple manipulations with (4.12) yield $\partial_t\,(\partial_s u - p\,\partial_s x - q\,\partial_s y) = -H_u\,(\partial_s u - p\,\partial_s x - q\,\partial_s y)$. It follows that $\partial_s u = p\,\partial_s x + q\,\partial_s y$ for all s and t, while (clearly) $\partial_t u = p\,\partial_t x + q\,\partial_t y$ also applies. Hence $p = u_x$ and $q = u_y$. **QED**

4.1.3 Examples.

Example 4.1 *The* **Eikonal** *equation. Take* $H = \dfrac{1}{2}\,\left(p^2 + q^2 - 1\right)$. *Then the equation is*

$$0 = \frac{1}{2}\,\left((u_x)^2 + (u_y)^2 - 1\right). \tag{4.14}$$

The characteristic equations are $\dot{x} = p,\ \dot{y} = q,\ \dot{u} = p^2 + q^2 = 1,$ *and* $\dot{p} = \dot{q} = 0$.

We now ask the following question: Does the equation allow solutions where $\boldsymbol{\nabla u}$ has a simple jump discontinuity along some curve? *As in (4.6), let* $\{\eta = \eta(x,\, y),\ \xi = \xi(x,\, y)\}$ *be the coordinate functions for a (local) orthogonal curvilinear coordinate system such that* ∇u *has a simple jump discontinuity along some curve* $\eta =$ *constant. In the new coordinates the equation is*

$$\begin{aligned} 0 &= \frac{1}{2}\,\left((\eta_x\,u_\eta + \xi_x\,u_\xi)^2 + (\eta_y\,u_\eta + \xi_y\,u_\xi)^2 - 1\right) \\ &= \frac{1}{2}\,\left((\eta_x^2 + \eta_y^2)\,(u_\eta)^2 + (\xi_x^2 + \xi_y^2)\,(u_\xi)^2 - 1\right). \end{aligned} \tag{4.15}$$

Thus, for equation (4.14), it should be clear that:

> Curves across which ∇u has a simple discontinuity are (in principle) allowed provided that the jump in the normal derivative[11] is in the form of a sign change. (4.16)

These curves are not characteristics (the characteristic equations are not even defined on them).

As an **example,** *in the domain* $-1 < x < 1$, *let* $u = u^\dagger = 1 - |x|$. *This is an exact solution to equation (4.14), for which* u_x *is discontinuous along the line* $x = 0$ — *note that* $u^\dagger_x = -sign(x)$ *and* $u^\dagger_y = 0$. *The characteristics are the two family of (horizontal) lines, terminating at the discontinuity:*
(f_L) $x = t - 1,\ y = y_0,\ u = t,\ p = +1$ *and* $q = 0$, *where* y_0 *is arbitrary and* $0 \le t \le 1$.
(f_R) $x = 1 - t,\ y = y_0,\ u = t,\ p = -1$ *and* $q = 0$, *where* y_0 *is arbitrary and* $0 \le t \le 1$.

---
[10] In this argument ∂_s (resp. ∂_t) indicates the derivative with respect to s (resp. t) keeping t (resp. s) constant.
[11] There can be no jump in the tangential derivative, assuming a nice enough curve, because u is continuous.

This solution $u^\dagger$ can be interpreted as the distance from the boundary in the domain $-1 < x < 1$, where the singularity occurs at the locus of the points where there is a switch in which of the two sides of the boundary is closest. The line $x = 0$ is a shock, *not a characteristic. As we show below,* a restriction — in addition to (4.16) — needs to be imposed on the solutions *to avoid non-uniqueness issues. The analog of the* entropy condition *for shocks in Gas Dynamics.*

In general, **distance functions**[12] **solve (4.14).** *For such functions, discontinuities in ∇u occur at the points that are equidistant from several points on the set from which the distance is being measured. Vice versa, any solution of (4.14) can be interpreted (at least locally) as being a distance function.*[13] *With this interpretation in mind, it should be clear that a restriction on the discontinuities that satisfy (4.16) must be imposed, as follows:*[14]

> Discontinuities in ∇u, of the type in (4.16), are admissible if $u\,\mathrm{sign}(u)$ has a local maximum at the curve of the discontinuity, along the directions normal to the curve. (4.17)

To see why this is needed, consider the following 1-D simple version of the problem: *solve* $u_x^2 = 1$ *for* $-1 < x < 1$, *with* $u = 0$ *at* $x = \pm 1$. *This has no solution unless discontinuities in u_x, of the type in (4.16), are allowed. Then, if in addition (4.17) is imposed, there are two solutions $u = \pm(1 - |x|)$ — a unique solution follows if we require a non-negative answer (distance function). On the other hand, if (4.17) is not imposed, it is easy to see that infinitely many solutions are possible.*

Another **example of a distance function solution of (4.14)** *is given by*

$$u = \min_{\pm} \sqrt{(x \pm 1)^2 + (y \pm 1)^2}. \tag{4.18}$$

This solution has shocks along the coordinate axes.

Finally, a **solution of the type described in (4.6) through (4.11)** *is given by*

$$u = \sqrt{x^2 + y^2} \quad \text{for } y \geq 0, \qquad \text{and} \qquad u = x \quad \text{for } y \leq 0. \tag{4.19}$$

Note that this is the distance to the negative y-axis. For this solution u_{yy} is discontinuous along the two characteristic curves: $x = \pm t$, $y = 0$, $u = t$, $p = \pm 1$, *and* $q = 0$ — *where* $t > 0$.

Example 4.2 *The* **quasi-linear** *equation case. Take $H = a\,q + b\,p - c$, where a, b, and c, are functions of $(u,\, x,\, y)$. Then the equation is*

$$a\,u_y + b\,u_x = c, \tag{4.20}$$

with characteristics $\dot{x} = b$, $\dot{y} = a$, $\dot{u} = b\,p + a\,q = c$, etc. This case is an **exception to the "generic" arguments in remarks 4.2 and 4.3.** *First of all, in order to guarantee existence and uniqueness for*

[12]That is, u is the distance from some set. For example a curve, or a collection of curves and isolated points.
[13]Possibly modulo a transformation of the form $u \to \pm u$+constant, under which the solutions are invariant.
[14]Distance functions are non-negative, but here we consider also the possibility of a signed distance function.

the solutions of (4.4), it is enough if u *is Lipschitz continuous (no differentiability needed). Second, the condition* $H = 0$ *does not restrict the size of the jumps allowed in* ∇u. *In other words, weak singularities in the solutions involving* ∇u *are allowed:* ∇u *can have an unrestricted simple jump discontinuity in its component normal to the characteristics, across the characteristics. However, when this happens, the sub-equations (4) and (5) in (4.12) have to be interpreted carefully: they apply on each "side" of the characteristic, and they carry the corresponding value of* ∇u.

Finally, we recall that: if (4.20) can be obtained from a conservation law, then solutions with shocks for it may make sense (or not, depending on the nature of what is conserved).

Example 4.3 *Take* $H = q + \frac{1}{2}\,p^2$. *Then the equation is*

$$u_y + \frac{1}{2}\,u_x^2 = 0, \tag{4.21}$$

with characteristics $\dot{x} = p$, $\dot{y} = 1$, $\dot{u} = q + p^2 = \frac{1}{2}\,p^2$, *and* $\dot{p} = \dot{q} = 0$. *The analog of equation (4.15) is then*

$$\begin{aligned} 0 &= \eta_y\,u_\eta + \xi_y\,u_\xi + \frac{1}{2}\,(\eta_x\,u_\eta + \xi_x\,u_\xi)^2 \\ &= \frac{1}{2}\,\eta_x^2\,u_\eta^2 + (\eta_y + \eta_x\,\xi_x\,u_\xi)\,u_\eta + \xi_y\,u_\xi + \frac{1}{2}\,\xi_x^2\,u_\xi^2. \end{aligned} \tag{4.22}$$

Thus a discontinuity in ∇u *can occur across a curve, provided that the jump in the derivative of* u *normal to the curve,*[15] *across the curve, satisfies*

$$0 = \frac{1}{2}\,\eta_x^2\,[u_\eta^2] + (\eta_y + \eta_x\,\xi_x\,u_\xi)\,[u_\eta], \tag{4.23}$$

where the brackets $[\cdot]$ *denote the jump in the enclosed quantity. Just as in the case of example 4.1, the curves where this happens are "shocks", not characteristics — and additional restrictions are needed to make the solutions unique. The issue of whether or not solutions with this type of discontinuity depends on the physical context in which equation (4.21) occurs.*

In particular, consider the situation where $\boldsymbol{y = t =}$ time, *and solutions with discontinuous derivatives are allowed. Then* causality provides the extra condition that shocks must satisfy: *the characteristics must converge into the shock path as time advances. Next we write the shock conditions in a more familiar form:*

(i) *From* $u_x = \eta_x\,u_\eta + \xi_x\,u_\xi$ *and* $u_y = \eta_y\,u_\eta + \xi_y\,u_\xi$ *it follows that:*
$[u_x] = \eta_x\,[u_\eta]$, $[u_y] = \eta_y\,[u_\eta]$, *and* $[u_x^2] = \eta_x^2\,[u_\eta^2] + 2\,\eta_x\,\xi_x\,u_\xi\,[u_\eta]$.

(ii) *Using (i), equation (4.23) takes the (not too surprising) form:* $[u_y] + \frac{1}{2}\,[u_x^2] = 0$.

[15] There can be no jump in the tangential derivative, assuming a nice enough curve, because u is continuous.

(iii) The shock velocity *is given by* $s = \dfrac{dx}{dy} = -\dfrac{\eta_y}{\eta_x}$. *Hence, from (i),* $[u_y] = -s\,[u_x]$.

Thus, using (ii) and (iii) we can write

$$-s\,[u_x] + \frac{1}{2}\,[u_x^2] = 0 \quad\Longrightarrow\quad s = \text{average}(u_x), \tag{4.24}$$

along shocks. Since the characteristics are given by $\dot{x} = u_x$, causality becomes: $\boldsymbol{u_x}$ decreases across the shock.

Example 4.4 *Take* $H = q + p^2 + u$, *and solve the problem with data* $u(x,\,0) = x$. *The equation is*

$$u_y + u_x^2 + u = 0, \qquad \textit{with} \tag{4.25}$$

1. *Characteristics:* $\dot{x} = 2\,p$, $\dot{y} = 1$, $\dot{u} = q + 2\,p^2 = p^2 - u$, $\dot{p} = -p$, *and* $\dot{q} = -q$.
2. *From the data:* $x = s$, $y = 0$, $u = s$, $p = 1$, *and* $q = -1 - s$, *on the characteristics at* $t = 0$ *— where* s *is a parameter* $-\infty < s < \infty$.
3. *It follows that:* $x = 2 + s - 2\,e^{-t}$, $y = t$, $u = (1+s)\,e^{-t} - e^{-2t}$, $p = e^{-t}$, *and* $q = -(1+s)\,e^{-t}$.

Hence, solving for t *and* s *in terms of* x *and* y *— that is:* $t = y$ *and* $s = x - 2 + 2\,e^{-y}$, *and substituting this into the expression for* u *in item 3 above, we obtain:*

$$u = (x-1)\,e^{-y} + e^{-2\,y}, \quad u_x = p = e^{-y}, \quad \textit{and} \quad u_y = q = (1-x)\,e^{-y} - 2\,e^{-2\,y}. \tag{4.26}$$

Note that the "characteristic" map $(s,\,t) \to (x,\,y)$ *is one-to-one and onto. Hence* **the solution is defined uniquely and everywhere by the equation and the data.**

4.2 Characteristics for the general first order scalar pde in n-D.

The most general scalar first order pde in n-D can be written in the form

$$H(u,\,\vec{p},\,\vec{x}) = 0, \quad \text{where } \; p_j = \frac{\partial u}{\partial x_j} \;\text{ for }\; 1 \le j \le n. \tag{4.27}$$

Here H is some given function, which we assume is twice continuously differentiable.

4.2.1 Strips and Monge cones. Not yet done.

4.2.2 Characteristics for the n-D first order scalar pde. Not yet done.

4.2.3 Characteristics and Hamiltonian Systems. Not yet done.

4.2.4 The Hamilton-Jacobi equation $u_t + H(\nabla u,\,\vec{x}) = 0$. Not yet done.

5 Boundary layers.

In this section we include some examples of boundary layer calculations.

5.1 Advection-diffusion in 1-D.

Consider the problem

$$u_t = \epsilon\, u_{x\,x} + u_x \quad \text{for } \; x > 0, \tag{5.1}$$

with the *boundary condition:*

$$0 = \epsilon\, u_x(0,\, t) + u(0,\, t), \tag{5.2}$$

and *initial condition:*

$$u(x,\, 0) = f(x), \tag{5.3}$$

where $0 < \epsilon \ll 1$. This problem is a nondimensional version of the problem introduced in § 5.1.1.

Since $0 < \epsilon \ll 1$, it is natural,[16] to expect that the solution can be approximated by an expansion of the form

$$u \sim u_0(x,\, t) + \epsilon\, u_1(x,\, t) + \epsilon^2\, u_2(x,\, t) + \dots \tag{5.4}$$

Substituting this into (5.1), successively solving for each order in ϵ, and enforcing the initial condition, yields

$$u_0 = f(x+t), \;\; u_1 = t\, f''(x+t), \;\; \dots \;\; u_n = \frac{1}{n!}\, t^n\, f^{(2n)}(x+t), \;\; \dots \tag{5.5}$$

That is, formally:

$$u \sim \sum_{n=0}^{\infty} \frac{1}{n!}\, \epsilon^n\, t^n\, \partial_x^{2n}\, f(x+t) = e^{\epsilon\, t\, \partial_x^2}\, f(x+t). \tag{5.6}$$

However, *this does not satisfy the boundary condition (5.2),* the reason being that the expansion in (5.4) ignores the fact that,[17] in order to stop the flux at the origin, large gradients are generated there. Below we show how to correctly approximate the solution near the origin.

Remark 5.1 *In (5.4) and (5.6) we use the symbol $\sim$ because the series involved do not, in general, converge. In addition, we expect them to approximate the solution in an asymptotic sense only: the error in adding up the series up to some term is of the same order/size as the first neglected term. Further, in the particular case of (5.4) and (5.6) this should apply only if one does not use the series too close to the origin. How close is too close to follows from the expansion below, i.e.: $\epsilon \ll x$ is needed — see the arguments above equation (5.13).*

In order to capture the behavior near the origin, we change variables to introduce the scale on which diffusion can balance advection. The problem then becomes

$$\epsilon\, u_t = u_{\xi\,\xi} + u_\xi \quad \text{for } \; \xi = \frac{x}{\epsilon} > 0, \tag{5.7}$$

with the *boundary condition:*

$$0 = u_\xi(0,\, t) + u(0,\, t). \tag{5.8}$$

For now we ignore the initial condition — note that f in (5.3) has no dependence on ξ, as we will

[16] Albeit incorrect, as we will soon see.

[17] See remark 5.3.

see, *the boundary layer is produced by the time evolution and the boundary condition,* not the initial conditions. Next we expand

$$u \sim \frac{1}{\epsilon}\,\tilde{u}_0(\xi,\,t) + \tilde{u}_1(\xi,\,t) + \epsilon\,\tilde{u}_2(\xi,\,t) + \dots \tag{5.9}$$

Note: For the gradients to be large near the origin, the solution itself must be large there. Hence the ϵ^{-1} leading order here. Physically: the particles drift to the bottom, and accumulate there, till a large enough gradient is generated that stops further accumulation. See remark 5.3.

Substituting (5.9) into (5.7) and (5.8), and collecting equal orders in ϵ, yields

$$(\tilde{u}_0)_{\xi\,\xi} + (\tilde{u}_0)_\xi = 0 \ \text{ for } \ \xi > 0, \quad \text{with} \quad (\tilde{u}_0)_\xi + \tilde{u}_0 = 0 \ \text{ at } \ \xi = 0, \tag{5.10}$$

and

$$(\tilde{u}_n)_{\xi\,\xi} + (\tilde{u}_n)_\xi = (\tilde{u}_{n-1})_t \ \text{ for } \ \xi > 0, \quad \text{with} \quad (\tilde{u}_n)_\xi + \tilde{u}_n = 0 \ \text{ at } \ \xi = 0, \tag{5.11}$$

for $n \geq 1$. Thus

$$\begin{aligned}
\tilde{u}_0 &= a_0(t)\,e^{-\xi}, \\
\tilde{u}_1 &= \left(a_1(t) - \xi\,\dot{a}_0\right)\,e^{-\xi} + \dot{a}_0, \\
\tilde{u}_2 &= \left(\dot{a}_1 - 2\,\ddot{a}_0 + \xi\,\ddot{a}_0\right) + \left(a_2 + \xi\,\left(\ddot{a}_0 - \dot{a}_1\right) + \tfrac{1}{2}\,\xi^2\,\ddot{a}_0\right)\,e^{-\xi}, \quad \dots
\end{aligned} \tag{5.12}$$

where the a_n are some functions of time.

The expansion in (5.9) must *match* the one in (5.4) as x moves away from the origin. This determines the functions a_n. Specifically, **consider values of x such that $\epsilon \ll x \ll 1$,** where we expect both expansions to apply.[18] In this limit (5.4) reduces to

$$u \sim \left(f(t) + x\,f'(t) + \dots\right) + \epsilon\,\left(t\,f''(t) + x\,t\,f'''(t) + \dots\right) + \dots \tag{5.13}$$

while (5.9) gives[19]

$$u \sim \left(\dot{a}_0 + x\,\ddot{a}_0 + \dots\right) + \epsilon\,\left(\dot{a}_1 - 2\,\ddot{a}_0 + \dots\right) + \dots \tag{5.14}$$

It follows that

$$\boldsymbol{\dot{a}_0 = f(t), \quad \dot{a}_1 = \frac{d^2}{dt^2}\left(t\,f(t)\right), \quad \dots} \tag{5.15}$$

with initial conditions:[20] $\boldsymbol{a_0(0) = 0}$, $\boldsymbol{a_1(0) = 0}$, etc.

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[18] You may have noticed that there are many "we expect" statements in this section. Proving these things rigorously, even for a simple linear problem such as the one here, is not trivial. Generally one has to be satisfied with verifying that everything fits consistently — though without assuming consistency, i.e.: do as many sanity checks as possible.

[19] All the $e^{-\xi}$ terms associated with the boundary layer do not contribute here, as they are exponentially small.

[20] The expansion in (5.9) must yield $u(0,\,0) = f(0)$.

Note that the time evolution for a_0 guarantees that the boundary layer, represented at leading order by $u \approx \dfrac{1}{\epsilon}\, a_0(t)\, e^{-\xi}$, grows in such a way that u is conserved. This follows because the total amount of "stuff" in the boundary layer $\approx \int \frac{1}{\epsilon}\, \tilde{u}_0\, dx = a_0$, must vary according to the influx $\approx f(t)$.

An approximation to the solution, valid for all x, can be obtained by combining the two expansions, as follows:

$$u \sim \frac{1}{\epsilon} \int_0^t f(s)\, ds\, e^{-\xi} + \left(t\, f'(t) + f(t) - f(0) - \xi\, f(t)\right)\, e^{-\xi} + f(x+t). \tag{5.16}$$

Remark 5.2 *Note that this approximation is (generally) not good for large times, specifically: it is valid only as long as $\epsilon\, t \ll 1$. This can be seen from equation (5.6), which shows that the higher order "corrections" are not small unless $\epsilon\, t \ll 1$.*

The reason for this is that the dissipation in the equation (even though small in the absence of large gradients) still has an effect over long time scales: essentially, it "flattens" the solution, eliminating variations. Hence, as initially present (far from the origin) "bumps and valleys" in the initial conditions move towards the origin, they are slowly "erased" from the solution profile. The expansions here do not account for this slow time evolution, hence they fail once its effect becomes important.

To fix this problem one could, in principle, modify the expansions so as to incorporate the slow time scale $\epsilon\, t$. Unfortunately, this leads to asymptotic equations which are, essentially, equivalent to the full problem — that is: no simplification occurs. The reason is that the problem in (5.1 – 5.3) is already the simplest possible problem in terms of diffusion effects in the bulk, no further simplifications are possible without loosing the effect.

On the other hand: if the derivatives of f decay fast enough at infinity (that is, f is already "flat" at infinity), then there is no problem with the expansion for large times. *For example, assume that $f^{(n)}(s) = O(s^{-n+\alpha})$ for $s \gg 1$, for some constant α. Then the combination $t^n\, f^{(2n)}(t+x)$ in (5.5) is never large, even if $t \gg 1$.*

5.1.1 Motivation/Example.

Consider the situation where *small particles in a liquid column of depth L are settling under the influence of gravity, while undergoing brownian motion.* We will now make a very simple model for this (I stress the "simple" in simple model, do not take it too seriously).

Let $0 < x < L$ be the depth coordinate, with the surface at $x = L$ and the bottom at $x = 0$. If the particles are very small, inertia is not important, and their motion is determined by the balance between the gravitational force on each particle, and the fluid drag. In addition, if the particles are not too close together,[21] this leads to a constant downward velocity — which creates a flux $\vec{q}_s = -a\, u\, \hat{n}$, where $\boldsymbol{u}$ is the *particle density,* $\boldsymbol{a > 0}$ is the *settling velocity*, and $\hat{n}$ is the unit vector

[21] Hence we can ignore their interactions via the fluid.

pointing up. The brownian motion creates an additional flux $\vec{q}_b = -\nu\,\mathrm{grad}(u)$, where $\boldsymbol{\nu > 0}$ is the *diffusivity*. Thus we obtain an *advection-diffusion* equation, which in 1-D is

$$u_t = \nu\, u_{x\,x} + a\, u_x \quad \text{for} \;\; x > 0, \tag{5.17}$$

where *the total flux is* $\boldsymbol{q = -\nu\, u_x - a\, u}$. In addition, we impose the *boundary conditions:*

— The flux vanishes at the bottom, $\quad 0 = q(0,\,t) = -\nu\, u_x(0,\,t) - a\, u(0,\,t). \quad$ (5.18)

— The flux vanishes at the surface, $\quad 0 = q(L,\,t) = -\nu\, u_x(L,\,t) - a\, u(L,\,t). \quad$ (5.19)

Remark 5.3 *In the absence of diffusion ($\nu = 0$), the solutions to (5.17) take the simple form $u = f(x + a\,t)$, which results because all the particles are going down at the same velocity. In this case the boundary condition in (5.18) does not make sense, since it leads to $0 = f(a\,t)$ for all times. That is, only $u \equiv 0$ satisfies it! The reason for this problem is clear: there is nothing in the equation that can stop the particles as they approach the bottom, so how can $q = 0$ happen there?*

A reasonable modeling assumption when $\nu = 0$ (particles are too large for the brownian motion to matter) is to posit that, as they arrive to the bottom, they stop and accumulate there at some maximum density u_{max}, with the effective position of the bottom moving up as particles arrive. If $x = \sigma(t)$ is this position (accumulation front), conservation of particles leads to the equation

$$\frac{d\sigma}{dt} = a\,\frac{u(\sigma,\,t)}{u_{\mathsf{max}} - u(\sigma,\,t)}, \quad \textit{where} \quad u(\sigma,\,t) = f(\sigma + a\,t) \tag{5.20}$$

and $u = f(x + a\,t)$ for $x > \sigma$. Of course, we must also assume that $0 \le f < u_{\mathsf{max}}$ — which is OK, since the "constant speed downward" assumption requires $u \ll u_{\mathsf{max}}$ anyway.

The problem above does not arise when $\nu > 0$, because then the diffusion term $\nu\, u_{x\,x}$ can balance the advection term $a\, u_x$ — provided that the density gradient is large enough near the bottom. When $0 < \nu \ll a\,L$ the needed gradient is large.