Hilbert Transform extras

(1) Integral formula for the Hilbert Transform on the unit circle ——— HT01

Here we first write the Hilbert Transform using Fourier Series and the definition, and then recast the answer in terms of integrals involving principal values.

Note 2 ——— HT03

Connection with the residue of the integral kernel at it's simple pole.

(2) Hilbert Transform for a generic simple closed curve ——— HT04

Here we begin with the problem of constructing an analytic function inside the curve, with prescribed real part on the curve and then use this to define the Hilbert Transform.

Integral formula for Hilbert transform on unit circle

Let $\varphi = \varphi(\theta)$ 2π-periodic, real valued and cont. diff. C^1

Let ~~The~~ $\psi = H\varphi$. (H.1)

<u>How do we get ψ?</u>

$$\varphi = \sum_{-\infty}^{\infty} \varphi_n e^{in\theta} \; ; \quad \varphi_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta) e^{-in\theta}\, d\theta \qquad (H.2)$$

Fourier Series

Since φ is real valued, $\varphi_{-n} = \overline{\varphi_n}$; thus

$$\varphi = \varphi_0 + 2\,\mathrm{Re}\left(\sum_1^{\infty} \varphi_n e^{in\theta}\right) \qquad (H.3)$$

Let

$$f = \varphi_0 + 2\sum_1^{\infty} \varphi_n z^n \qquad (H.4)$$

analytic $|z|<1$

Easy to see then, by definition, that

$$\psi = H\varphi = 2\,\mathrm{Im}\left(\sum_1^{\infty} \varphi_n e^{in\theta}\right) \qquad (H.5)$$

since

$$\lim_{|z|\to 1} f = \varphi + i\psi \qquad (H.6)$$

Note 1: ψ has mean zero, which is the restriction we need to impose to define ψ uniquely — see (H.18)

Next we substitute the expression for the φ_n in (H.2) (and switch Σ and $\int$) into (H.4).

Hence
$$f = \varphi_0 + \frac{1}{\pi}\int_0^{2\pi}\left\{\sum_1^{\infty}(ze^{-i\mu})^n\right\}\varphi(\mu)\,d\mu$$
$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{ze^{-i\mu}}{1-ze^{-i\mu}}+\frac{1}{2}\right)\varphi(\mu)\,d\mu$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\mu}+z}{e^{i\mu}-z}\,\varphi(\mu)\,d\mu \qquad \text{(H.7)}$$

Equivalently, with $\zeta = e^{i\mu}$

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{\zeta+z}{(\zeta-z)\,\zeta}\,\varphi\,d\zeta \qquad \text{(H.8)}$$

Then, with $\zeta_0 = e^{i\mu_0}$, See note 2.

$$\lim_{z\to\zeta_0} f(z) = \varphi(\mu_0) + \frac{1}{2\pi i}\int\frac{\zeta+\zeta_0}{(\zeta-\zeta_0)\,\zeta}\,\varphi\,d\zeta$$
$$= \varphi(\mu_0) + \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\mu}+e^{i\mu_0}}{e^{i\mu}-e^{i\mu_0}}\,\varphi(\mu)\,d\mu$$
$$= \varphi(\mu_0) + \frac{1}{2\pi i}\int_0^{2\pi}\frac{\cos\alpha}{\sin\alpha}\,\varphi(\mu)\,d\mu \qquad \text{(H.9)}$$

where $\boxed{\alpha = \frac{1}{2}(\mu-\mu_0)}$.

Thus

$$\boxed{\underbrace{\psi}_{\psi(\mu_0)} = H\varphi = \frac{-1}{2\pi}\int_0^{2\pi} \operatorname{cotan}(\alpha)\,\varphi(\mu)\,d\mu} \qquad \text{(H.10)}$$

Note 2 Why is the limit in (H.9) picking up $\varphi(\mu_0)$ plus a principal value?

The intuitive explanation in this limit ~~the answer should be~~ the principal value plus $\underbrace{\text{"(1/2) the residue at } z_0\text{"}}_{=\varphi(\mu_0)} \times 2\pi i$

The reason for the 1/2 is that the path goes only 1/2 around the pole at z_0.

Of course, since φ is not analytic we cannot use the residue theorem. However, the $\oint$ in (H.8) should yield

Hilbert Transform for generic simple closed curve

as long as φ is nice enough (e.g. C^1), a direct calculation shows that the answer is the same one would expect from residues.

Note 3: Where is the fact that φ has mean zero "buried" in (H.10)

This follows because $\cot(\alpha)$ is odd relative to μ $\therefore$ $\int \cot(\alpha)\, d\mu_0 = 0$ (H.11)

QUESTION Suppose that we have a "generic" simple closed curve Γ, positively oriented (counter-clockwise), which is the boundary of some simply connected open region Ω. Now let φ be a real valued function† defined on Γ. The question

(H.12)

† We assume φ and Γ are "nice enough"

is now: Is there an analytic function $f = f(z)$, defined in Ω, such that

$$\boxed{\lim_{z \to \Gamma} f = \varphi + i\psi} \qquad \text{(H.13)}$$

where ψ is also real?

The answer is yes: The Riemann mapping theorem says that there is an analytic bijection $z \longrightarrow w(z)$ from Ω to the unit disk, that preserves orientation. Using this we can map the problem into the unit disk and write

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{w(\zeta)+w(z)}{w(\zeta)-w(z)} \frac{w'(\zeta)}{w(\zeta)} \varphi(\zeta)\, d\zeta \qquad \text{(H.14)}$$

Note $\sim \dfrac{2}{\zeta - z}$ for ζ close to z #

Because of # the same calculation as in

(H.9) works so that, for $\xi_0 \in \Gamma$

$$\lim_{z \to \xi_0} f(z) = \varphi(\xi_0) + \frac{1}{2\pi i} \int_\Gamma \frac{W(\xi)+W(\xi_0)}{W(\xi)-W(\xi_0)} \frac{W'(\xi)}{W(\xi)} \varphi(\xi)\, d\xi \qquad \text{(H.15)}$$

Further, since $W(\xi) = e^{i\mu(\xi)}$ for $\xi \in \Gamma$ (where μ is real valued), we can write

$$\lim_{z \to \xi_0} f(z) = \varphi(\xi_0) + \frac{1}{2\pi i} \overbrace{\int_\Gamma \cot\!an(\alpha)\varphi(\xi)\underbrace{\mu'(\xi)\, d\xi}_{d\mu}}^{= i\tilde{\psi}} \qquad \text{(H.16)}$$

where $\alpha = \frac{1}{2}\mu(\xi) - \mu(\xi_0)$.

However we cannot conclude that

$$\tilde{\psi} = H\varphi$$

as we did in (H.10) from (H.9). The reason is that the zero mean property† does not carry over. That is:

(H.11) still applies, but what we

† see (H.18)

want/need is $\oint_{\Gamma} \cot\text{an}(\alpha)\, ds = 0$, while

$d\mu(\zeta_0) = |\mu'(\zeta_0)|\, ds$ and $|\mu'|$ is not a constant in general.

Thus we conclude $\boxed{\psi = H\varphi = \tilde{\psi} - \text{Mean}(\tilde{\psi})}$

H.17

Uniqueness

H.18

The problem posed in (H.12-13) defines f up to an additive pure imaginary constant. Thus, in order to pin-down f uniquely we require

$$\text{Mean}(\psi) = \frac{1}{L(\Gamma)} \int_{\Gamma} \psi\, ds = 0, \qquad \text{H.19}$$

where ds is the arclength and $L(\Gamma)$ is the length of Γ. Then

$$\psi = H\varphi \quad [\text{Hilbert Transform}]$$

is uniquely defined H.20

Proof Take any two solutions, f_1 and f_2, and write† $f_1 - f_2 = g = u + iv$. Then u is harmonic and solves the problem

$$\Delta u = 0 \text{ in } \Omega \ , \ u = 0 \text{ on } \Gamma$$

Thus, from the max-min principle $u \equiv 0$. From the Cauchy-Riemann equations

$$\nabla v \equiv 0 \text{ in } \Omega \Rightarrow v = \text{constant}. \quad QED$$

† Here $u = \mathrm{Re}(g)$, $v = \mathrm{Im}(g)$ are harmonic conjugates because g is analytic

Note 3: Is (H.16-17) "useful"? These formulas depend on the Riemann mapping $z \longrightarrow w$, which generally is not known and must be computed numerically (this is possible). If one needs to compute $H\varphi$ for many φ's this might be worth doing, as this reduces $H\varphi$ to a line integral (only the values of W on Γ are needed). However, even then (H.16) amounts to a full matrix multiplication from a discretized φ to a discretized ψ.

Thus it may turn out to be better to write $f = u + iv$, solve $\Delta u = 0$ in Ω with $u = \varphi$ on Γ; and then get v from† $\nabla v = (-u_y, u_x)$. (H.21)

$\boxed{\text{Thus (H.16-17) has more "theoretical" interest than (probably) practical "value"}}$ (H.22)

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† Note that we can get ψ from

$$\psi(s) = \psi_0 + \int_0^s (\nabla v)\cdot\vec{t}\, ds$$

s = arclength on Γ and $\vec{t}$ = unit tangent

Hilbert Transform inverse is $-H$.

Consider Ω and Γ as in (H.12) p HT.04, with φ a real valued function defined on Γ. Then $\boxed{H\psi = -\varphi + \mathrm{Mean}(\varphi)}$,

where $\psi = H\varphi$ (H.23)

and Mean is defined in (H.19) p HT.07.

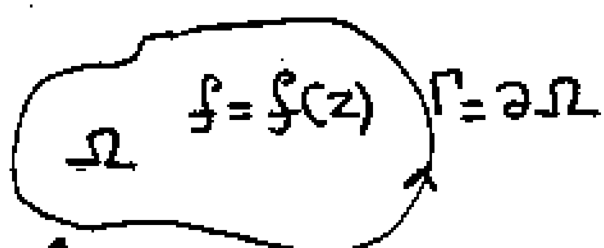
In particular † $\boxed{H^2 = -1}$ H.24

when restricted to mean zero functions

† Hence $H^{-1} = -H$ (H.24)

Proof

Recall the definition :



lim f on $\partial\Omega$

$f = \varphi + i\psi$

> Given a function φ (real valued) defined on Γ, find an analytic function in Ω, f, such that $\lim_{z\to\Gamma} \mathrm{Re}(f) = \varphi$ and $\lim_{z\to\Gamma} \mathrm{Im}(f) = \psi$ has zero mean. Then $\psi = H\varphi$

Now let $\underline{g = -if + i\,\mathrm{Mean}(\varphi)}$. Then: g is analytic in Ω and

(as $z \to \Gamma$) $g \to \psi + i(-\varphi + \mathrm{Mean}(\varphi))$.

Since $\mathrm{Mean}(-\varphi + \mathrm{Mean}(\varphi)) = 0$, $H\psi = -\varphi + \mathrm{Mean}(\varphi)$

QED

<u>Note</u> In lecture 14 we showed that: (i) for the unit circle, in terms of ~~the~~ Fourier Series, H is multiplication by $-i\,\mathrm{sig}(n)$; with $\mathrm{sign}(0) = 0$. This, of course agrees with (H.23). (ii) For the real axis H is multiplication by $-i\,\mathrm{sign}(k)$, which also agrees with (H.23).†

† For the real axis as Γ, and the upper half-plane as Ω, the mean zero condition is replaced by "$f(z)$ vanishes as $|z| \to \infty$" and so, for L^2 functions at least, $H^2 = -1$.