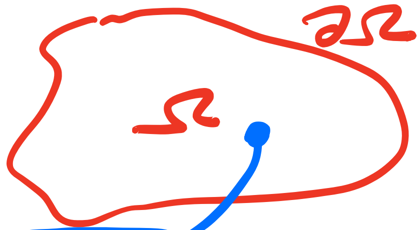
Lecture 24 Boundary Integral Method [BIM]



$\Delta\varphi = 0$

$$\boxed{\int_{\Omega} (\Delta\varphi)\,\psi = \int_{\partial\Omega} \varphi_n \psi - \int_{\partial\Omega} \varphi \psi_n + \int_{\Omega} \varphi \Delta\psi}$$

Case a: ψ = Green's function in R^d $= G$ Fundamental Sln Laplace

In 2-D $G = \frac{1}{2\pi} \ln|\vec{x}-\vec{y}|$

3D $G = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}$

$$0 = \int_{\partial\Omega} \varphi_n G - \int_{\partial\Omega} \varphi G_n + \varphi$$

Case b: $\psi = H$, $\Delta H = 0$ and $H_n = G_n$ on $\partial\Omega$

$$0 = \int_{\partial\Omega} \varphi_n H - \int_{\partial\Omega} \varphi G_n$$

$$\boxed{\mu = \varphi_n H/G - \varphi_n}$$

$$\boxed{\varphi = -\int_{\partial\Omega} \varphi_n G + \int_{\partial\Omega} \varphi_n H = \int_{\partial\Omega} \mu G}$$

μ = single layer potential

Same argument $\Rightarrow$

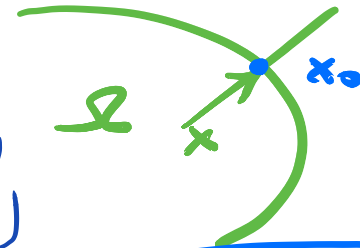
$$\boxed{\varphi = \int_{\partial\Omega} \rho\, G_n}$$

ρ = double layer potential

Suppose I want to solve $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_n = f & \text{on } \partial\Omega \end{cases}$

$u = \int_{\partial\Omega} \rho \, G_n$ $\quad \rho$?

$$u(x) = \int_{\partial\Omega} \rho(y)\, \hat{n}\cdot\nabla G(|x-y|)\,dy$$



$$\lim_{x\to x_0} u = \frac{1}{2}\rho(x_0) + \int_{\partial\Omega} \rho(y)\, G_n(x_0,y)\,dy$$

$$\boxed{f(x_0) = \frac{1}{2}\rho(x_0) + \int_{\partial\Omega} \rho(y)\, G_n(x_0,y)\,dy}$$

Solve integral equ. for $\rho \to$ soln

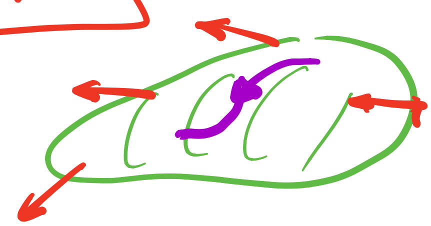
Incom. N.S.

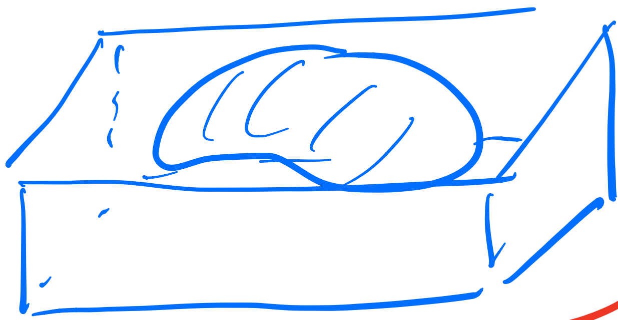
$$\vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho}\nabla p = \nu\Delta\vec{u}$$

$$\operatorname{div}\vec{u} = 0$$

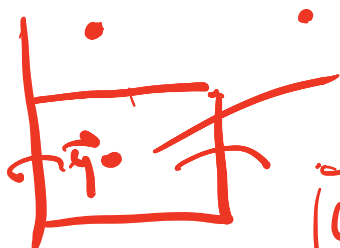
$$\boxed{\Delta p = -\operatorname{div}(\vec{u}\cdot\nabla\vec{u})}$$

proj. method





Laplace by images



$$\begin{bmatrix} \Delta u = 0 \\ u \text{ periodic} \end{bmatrix}$$

Want Green's function for this

$$G(\vec{x},\vec{y}) = \frac{1}{2\pi}\sum_{n,m} \ln|\vec{x}-\vec{y}_{nm}|$$

$\Delta G = \delta(\vec{x})$

$\Delta(G-\psi) = \delta(\vec{x})$

$\vec{y} = (y_1, y_2)$

$\vec{y}_{nm} = (y_1 + n p_1, y_2 + m p_2)$

$$\ln|\vec{x}-\vec{y}_{n,m}| - \ln\left|\sqrt{n^2+m^2}\right| - \alpha_n x - \beta_n y = O\left(\frac{1}{n^2+m^2}\right)$$

$$\ln|x-n| = \log\left[n\left(\frac{x}{n}-1\right)\right] = \log n + \log\left(1-\frac{x}{n}\right)$$

$$= \log n + \frac{x}{n} + \frac{x^2}{2n^2} + \cdots$$

$$G = \frac{1}{2\pi}\sum_{n,m}\Big[\log|x-y_{nm}| - \ln\left|\sqrt{n^2+m^2}\right| - \alpha_n x - \beta_n y\Big]$$

Boundary layers "Various Lecture notes 306"
pp 18-20

$$\boxed{u_t = +u_x + \epsilon u_{xx}}$$ $x>0$

$u(x,0) = f(x)$ and flux $=0$ at $x=0$

$$\boxed{0 = \epsilon u_x + u \text{ at } x=0}$$

$0 < \epsilon << 1$

Then at leading order let $\epsilon = 0$ $\quad u \simeq f(x+t)$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \quad \#1$$

$u_{0t} - u_{0x} = 0$

$u_{1t} - u_{1x} = u_{0xx}$

$u_{2t} - u_{2x} = u_{1xx}$

$= f(x+t) + \epsilon t f''(x+t)$
$+ \frac{\epsilon^2 t^2}{2} f^{iv}(x+t) \cdots$
$= \sum \frac{1}{n!} \epsilon^n t^n \partial_x^{2n} f(x+t)$

"Solution" near wall stretch coordinates

$$\boxed{\xi = \frac{x}{\epsilon}}$$

$$\boxed{\begin{aligned} \epsilon u_t &= u_{\xi\xi} + u_\xi, \text{ for } \xi > 0 \\ 0 &= u_\xi + u \quad \text{at } \xi = 0 \end{aligned}}$$

$U = U_0 + \epsilon U_1 + \cdots \quad \#2$

$\frac{1}{\epsilon}$

$u_{0\zeta\zeta} + u_{\zeta} = 0$ $\qquad a_0 e^{-\zeta} = u_0$ $\qquad a_0 = a(t)$

$u_{1\zeta\zeta} + u_{1\zeta} = (u_0)_t$ $\qquad u_1 = (a_1(t) - \zeta a_0) e^{-\zeta} + \dot{a}_0(t)$

$u_n = (\quad) e^{-\zeta} + (\quad)$

Such that $\boxed{\zeta \text{ large}}$

(b) x small

$$u \sim (\cancel{f(t)} + x\cancel{\dot{f}}(t) + \cdots) + \epsilon(\cancel{t\dot{f}} + xt\dddot{f} + \cdots)$$

$$u \sim (\cancel{\dot{a}_0} + x\cancel{\ddot{a}_0} + \cdots) + \epsilon(\cancel{\dot{a}_1} - 2\ddot{a}_0 \cdots)$$

$\boxed{\dot{a}_0 = f(t)}$ $\qquad \boxed{\dot{a}_1 = \frac{d^2}{dt^2}(tf)}$

Near wall $\qquad \boxed{u \sim \frac{1}{\epsilon}\left(\int^t f(s)\,ds\right) e^{-x/\epsilon}}$