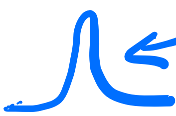
Lecture #23 May 18, 2021

Greens function heat eqn

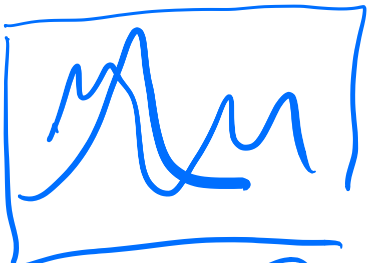
$u_t = u_{xx}, \ -\infty < x < \infty$

$G \propto \frac{1}{\sqrt{t}} e^{-\frac{(x-y)^2}{4t}}$

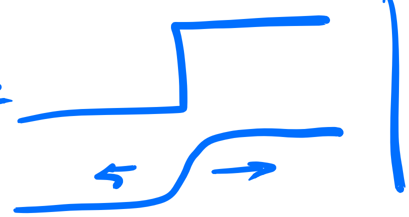
$u = \int G(x,y) f(y)\, dy$ $\quad \boxed{u(x,0) = f(x)}$



G its infinitly smooth and decay very fast as $|x| \to \infty$

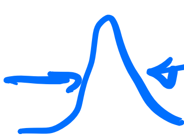


$f =$



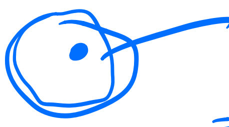
u

Error funct



$\sqrt{t}$

$\boxed{u_t = \nu \Delta u}$ $\quad \nu = \text{length}^2/\text{time}$



$\boxed{R \sim \sqrt{t\nu}}$

Normal modes

$\dot{\vec{Y}} = A\vec{Y}$ $\quad A\vec{v}_n = \lambda_n \vec{v}_n$

$Y = e^{\lambda_n t}\vec{v}_n$ is a solution

General sln $\quad \vec{Y} = \sum a_n e^{\lambda_n t}\vec{v}_n$

When? Example, yes!

A is self adjoint

Then λ_n real & $\vec{v}_n$ orthogonal

$$Y = \sum a_n e^{\lambda_n t} v_n \text{ where } \langle Y_0, \vec{v}_n \rangle = a_n \langle \vec{v}_n, \vec{v}_n \rangle$$

$Y_0 = Y(0)$ initial data

Can be generalized to "Normal"

$\underline{AA^+ = A^+A}$ commute with adjoint

Examples

A skew-adjoint, $A^+ = -A$ imaginary eigenvalues orthogonal eigenvectors

A unitary $A^+ = A^{-1}$ eigenvalues are on unit circle orthogonal eigenvectors

What happens for "operators"

Linear map from $H \xrightarrow{L} H$

H = Hilbert space (e.g. square integrable function somewhere)

$$\langle f, g \rangle = \int_\Omega f(\vec{x}) \bar{g}(\vec{x}) \mu(\vec{x}) d\vec{x} \qquad \boxed{\mu > 0}$$

Then self-adjoint means $\langle Af, g \rangle = \langle f, Ag \rangle$

and "modulo math. subtleties"
theorem above applies

$\therefore$ can write $\boxed{u_t = \mathcal{L}u \text{ using normal modes}}$

$$u_t = \sum a_n e^{\lambda_n t} \varphi_n \quad \text{where } \mathcal{L}\varphi_n = \lambda_n \varphi_n$$

Example Heat eqn periodic B.C.

$u_t = u_{xx}$ periodic period 2π

it is self adjoint $\int_0^{2\pi} f g'' dx = \int_0^{2\pi} f'' g \, dx$

f, g periodic

$$\boxed{\varphi_n'' = \lambda \varphi_n}$$ periodic period 2π

$\varphi_n = e^{inx}$ n integer

$\lambda_n = -n^2$

$$u = \sum_{-\infty}^{\infty} a_n e^{inx - n^2 t} \qquad u|_{t=0} = f$$

$$\langle f, \varphi_n \rangle = a_n \underbrace{\langle \varphi_n, \varphi_n \rangle}_{2\pi}$$

Fourier series

2π $\quad inx$

$$a_n = \frac{1}{2n}\int_0 f(x)\, e^{-inx}\, dx$$

Example 2 $u_t = u_{xx}$, $0 < x < \pi$

$u\big|_{x=0} = 0$, $u_x\big|_{x=\pi} = 0$

$\phi_n'' = \lambda_n \phi_n$ $\quad \phi(0) = 0 \quad \phi'(\pi) = 0$

$\phi_n = \sin[(n+\frac{1}{2})x] \quad \lambda_n = -(n+\frac{1}{2})^2$

$$\boxed{u = \sum a_n e^{-\lambda_n t} \sin[(n+\tfrac{1}{2})x]}$$

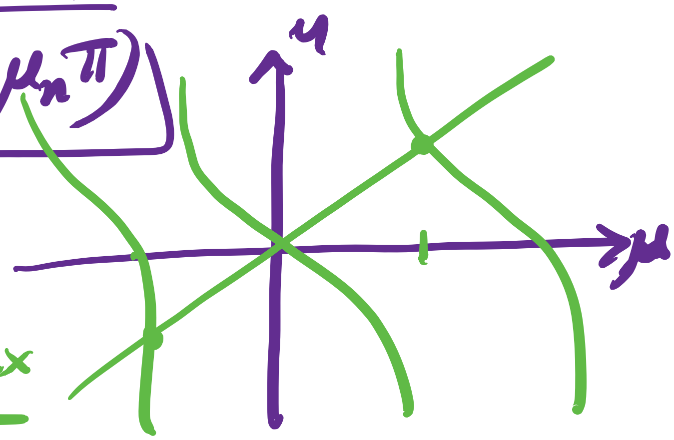
Ex. 3 $u_t = u_{xx}$, $0 < x < \pi$

$u\big|_{x=0} = 0$, $u + \alpha u_x \big|_{x=\pi} = 0$

$$\boxed{\phi_n = \sin(\mu_n x)}$$

$\sin(\mu_n \pi) + \alpha \mu_n \cos(\mu_n \pi) = 0$

$$\boxed{\alpha \mu_n = -\tan(\mu_n \pi)}$$



$u = \sum a_n e^{\lambda_n t} \sin(\mu_n x)$

$$a_n = \frac{\int_0^\pi f(x) \sin(\mu_n x)\, dx}{\int_0^\pi \sin^2(\mu_n x)\, dx}$$

Ex. #4 $\qquad u_t = u_{xx}\,,\quad -\infty < x < \infty$

cond. at ∞ $\quad$ u vanishes

$\boxed{\phi'' = \lambda\phi}$ bounded $\left[\begin{array}{l}\phi = e^{ikx} \\ \lambda = -k^2\end{array}\right|$ k real

"Generalized eigenfunctions"

$$\left[\begin{array}{l} u = \int \hat{u}(k)\, e^{ikx - k^2 t}\, dk \\ \hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx \end{array}\right.$$

Orthogonality

discrete $\quad \boxed{\langle \varphi_n, \varphi_m\rangle = \delta_{nm}\langle \varphi_n, \varphi_n\rangle}$

$$\int_{-\infty}^{\infty} e^{ikx}\, e^{-iKx}\, dx = 2\pi\,\delta(k-K)$$

Connection of normal modes to Green's functions $\qquad$ Ex. A

Heat eqn. on $-\infty < x < \infty$ $\qquad \boxed{u(x,0) = f(x)}$

$$u = \int_{-\infty}^{\infty} \hat{u}(k)\, e^{ikx - k^2 t}\, dk \;\Big|\; \hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} f(y)\, dy$$

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy\, e^{ik(x-y)-k^2 t} f(y) =$$

$$= \int_{-\infty}^{\infty} dy \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)-k^2 t} dk\right]}_{G(x,y,t)} f(y)$$

Green's function

$$G(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi - k^2 t} dk = \delta(\xi) \quad t=0$$

If $t>0$ $\quad K = \sqrt{t}\, k$

$$G = \frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{iK\left(\frac{\xi}{\sqrt{t}}\right) - K^2} dK$$

$$= \frac{C}{\sqrt{t}} e^{-\xi^2/4t}$$

Ex. B $\quad u_t = u_{xxx} \quad$ **Linear KdV**

$$u = \int_{-\infty}^{\infty} dk\, \hat{u}(k) e^{ikx - ik^3 t} \qquad \hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-iky} dy$$

$$u = \int^{\infty} dy \left[\frac{1}{2\pi}\int^{\infty} dk\, e^{ik(x-y)-ik^3 t}\right] f(y)$$

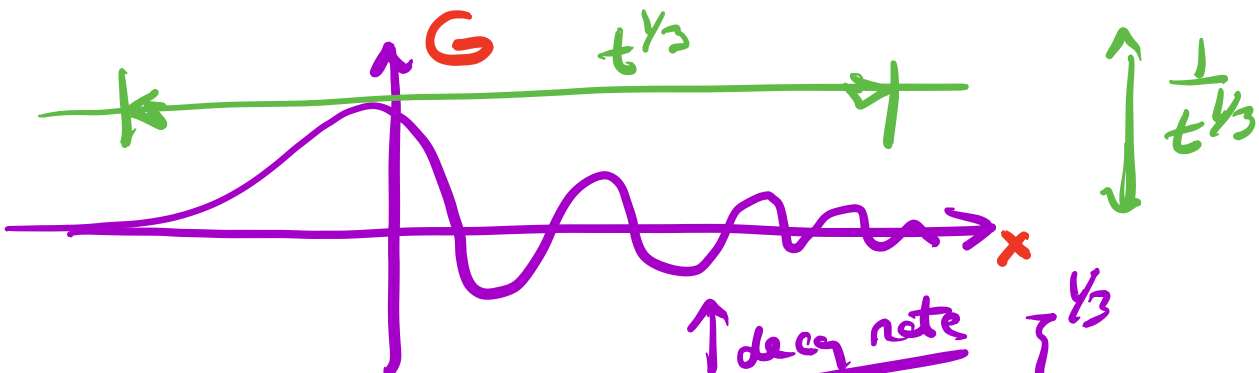
$$\underbrace{\qquad\qquad}_{G(x-y,t)}$$

$$G(\xi,t) = \frac{1}{2\pi t^{1/3}} \int_{-\infty}^{\infty} dK\, e^{iK\xi - iK^3}$$

$$\boxed{K = k t^{1/3}} \qquad \xi = \frac{x-y}{t^{1/3}}$$

$$= \frac{c}{t^{1/3}} \mathrm{Ai}\left[\frac{x-y}{t^{1/3}}\right]$$

Airy function



$$\boxed{u_t = u_{xxx}} \qquad e^{i(kx - \omega t)} \qquad \omega = k^3$$

phase velocity $c_p = k^2$ Group vel $\frac{1}{3}k^2$

Group velocity Side line

Slowly varying wavetrain

$$K = K(x,t)$$

$$\omega = \omega(x,t) = \Omega(K)$$

modulated
wave field

$$\boxed{\begin{array}{c} k_t + \omega_x = 0 \\ \omega = \Omega(k) \end{array}}$$

Ch. speed $\quad c_g = \frac{d\omega}{dk}$

If I add amplitude $\boxed{a = a(x,t)}$

$$\boxed{(a^2)_t + (c_g a^2)_x = 0}$$

Energy flux
velocity is c_g

Final example Discrete 4th

Have orthonormal bases ϕ_n

$u_t = \mathcal{L} u$

$$u = \sum_n a_n e^{\lambda_n t} \phi_n$$

$$a_n = \langle f, \phi_n \rangle$$

$$u = \sum_n e^{\lambda_n t} \phi_n \langle f, \phi_n \rangle$$

$$u(x,t) = \sum_n e^{\lambda_n t} \phi_n(x) \bar{\phi}_n(y) \mu(y) f(y)\, dy$$

$$= \int \underbrace{\left(\sum_n e^{\lambda_n t} \phi_n(x) \bar{\phi}_n(y) \mu(y) \right)}_{G} f(y)\, dy$$

$$\boxed{\mu(x) G(x,y) = \mu(y) \overline{G(y,x)}}$$

Next/last lecture

I will do BIM

Boundary Integral Methods

for Laplace

+ **Maybe** Convergence issue

Images

$$u_t = \mathcal{L}u = Au + Bu$$

dissipative part

$A = A^+$

$B = -B^+$ desir

$A = A^+ < 0$

Post lecture questions

Dispersive $u_t = \mathcal{L}u$ $\mathcal{L}$ = skew-adjoint

Dissipative $u_t = \mathcal{L}u$ $\mathcal{L} = \mathcal{L}^+ < 0$

$u_t = \mathcal{L}u$, $\mathcal{L}^+ = -\mathcal{L}$

$$\frac{d}{dt}\underbrace{\langle u, u\rangle}_{\int |u|^2 dx} = \langle u_t, u\rangle + \langle u, u_t\rangle =$$
$$= \langle \mathcal{L}u, u\rangle + \langle u, \mathcal{L}u\rangle =$$
$$= \langle \mathcal{L}u, u\rangle + \langle \mathcal{L}^+u, u\rangle = 0$$

$$i\varphi_t = \underbrace{\varphi_{xx} + V\varphi}_{} = \underbrace{(\partial_x^2 + V)}_{H}\varphi$$

$$\varphi_t = -iH\varphi \qquad \lambda = iE$$

$u_t = \mathcal{L}u$ $\mathcal{L} = \mathcal{L}^+ < 0$

$$\frac{d}{dt}\langle u, u\rangle = \langle u_t, u\rangle + \langle u, u_t\rangle =$$
$$= 2\langle \mathcal{L}u, u\rangle < 0$$

$$u_t = Au + Bu$$

$$\frac{d}{dt}\langle u,u\rangle = 2\langle Au,u\rangle +$$

$$\boxed{\dot{Y} = AY}$$

$$e^A = \sum_0^\infty \frac{1}{n!}A^n$$

$$\boxed{Y = e^{At}\cdot Y(0)}$$

$$\frac{d}{dt}e^{At} = Ae^{At}$$

$$\vec{u} = \sum a_n \varphi_n$$

$$A = \sum \lambda_n \varphi_n \varphi_n^T$$

$$\boxed{A\vec{u} = \sum \lambda_n a_n \varphi_n}$$

$$\boxed{e^A = \sum e^{\lambda_n}\varphi_n\varphi_n^T}$$

$$e^{\partial_x^2}$$

$$e^{\partial_x^2} f = \int \hat{f}(k)e^{-k^2}dk$$

$$e^A = \sum \frac{1}{n!}A^n$$

$$A\varphi_n = \lambda\varphi_n$$

$$\boxed{\varphi_n \cdot \varphi_m = \delta_{nm}}$$

$$\boxed{\mathcal{L} = \sum e^{\lambda_n} \varphi_n \varphi_n^T}$$

$$\mathcal{L}\varphi_m = e^{\lambda_m} \varphi_m$$

$$e^{A} \varphi_m = \sum \frac{1}{n!} A^n \varphi_m$$

$$= \left(\sum \frac{1}{n!} \lambda^{n}_{m}\right) \varphi_m$$

$$= e^{\lambda_m} \varphi_m$$

$$\boxed{\sum \frac{1}{n!} A^n = \sum e^{\lambda_n} \varphi_n \varphi_n}$$

$$e^{At}$$

$$\boxed{\psi = e^{iHt} \varphi_0}$$