Lecture 22, May 13, 2021 Green's functions

Greens function IVP for $T_t = T_{xx}$, $-\infty < x < \infty$

$\boxed{G_t = G_{xx} \text{ with I.D. } G = \delta(x) \; \dagger}$

① If T is a solution then $T(\alpha x, \alpha^2 t)$ *
is a soln for any const.

② $\alpha \delta(\alpha x) = \delta(x)$

$\boxed{\begin{aligned} &\int \alpha \delta(\alpha x) f(x)\, dx \\ &= \int \delta(z) f(\tfrac{z}{\alpha})\, dz = f(0) \end{aligned}}$ *

$\dagger$ Extra condition G is bounded! Why?

$\alpha G(\alpha x, \alpha^2 t)$ solves same problem

Uniqueness of soln $\Rightarrow$ $\boxed{\alpha G(\alpha x, \alpha^2 t) = G(x,t)}$

Take $\alpha = \frac{1}{\sqrt{t_0}}$ $\quad \frac{1}{\sqrt{t_0}} G\left(\frac{x}{\sqrt{t_0}}, \frac{t}{t_0}\right) = G(x,t)$

$\therefore$ at $t = t_0$ $\quad \frac{1}{\sqrt{t_0}} G\left(\frac{x}{\sqrt{t_0}}, 1\right) = G(x, t_0)$

But t_0 arbit.

$$G(x,t) = \frac{1}{\sqrt{t}} \underbrace{G\left(\frac{x}{\sqrt{t}}, 1\right)}_{g(x/\sqrt{t})}$$

$$\boxed{G(x,t) = \frac{1}{\sqrt{t}}\, g\left(\frac{x}{\sqrt{t}}\right)} \qquad \boxed{\frac{x}{\sqrt{t}} = \xi}$$

Substitute into eqn.

$$\boxed{\frac{d^2}{d\xi^2}g + \underbrace{\frac{1}{2}\xi\frac{dg}{d\xi} + \frac{1}{2}g}_{\frac{d}{d\xi}\left(\frac{1}{2}\xi g\right)} = 0}$$

$$\boxed{\frac{dg}{d\xi} + \frac{1}{2}\xi g = a}$$

Same argument as above for symmetry $x \to -x$ shows $\left| \begin{array}{l} G \text{ even in } x \\ \Longrightarrow \\ g \text{ even in } \xi \end{array} \right.$

$$\Longrightarrow a = 0$$

$$g = c\, e^{-\frac{1}{4}\xi^2}$$

$$c = ?$$

$$G = \frac{c}{\sqrt{t}}\, e^{-\frac{x^2}{4t}}$$

$$\int_{-\infty}^{\infty} G\,dx = c\underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{4}\xi^2}\,d\xi}_{2\sqrt{\pi}} = 1$$

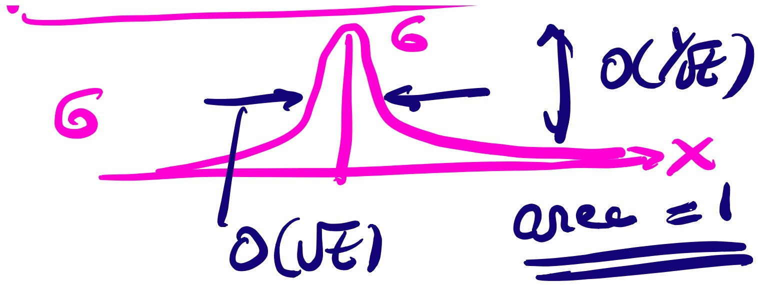
$$\xi = \frac{x}{\sqrt{t}}$$

$$c = \frac{1}{2\sqrt{\pi}}$$

$$G = \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t}$$

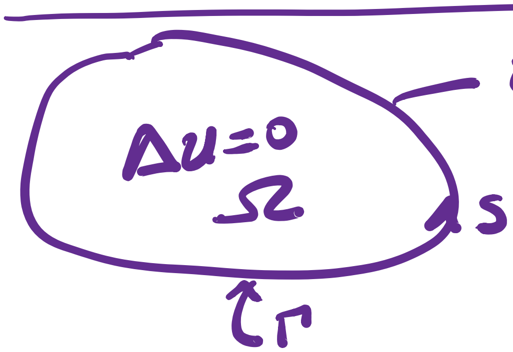
Green's Function Heat Eqn.

$t \downarrow 0$ $\quad G$



$$Ax = y \longrightarrow x = Gy \qquad G = A^{-1}$$

$$\sum_m a_{nm} x_m = y_n \qquad \underline{x_n = \sum G_{nm} y_m}$$



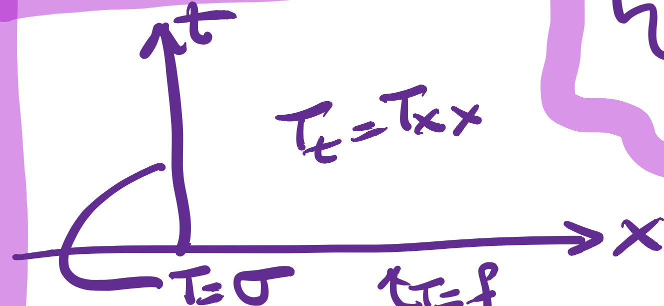
$\underline{u = f(s) \text{ on } \Gamma}$

solve

$\Delta G = 0$ in Ω

$G = \delta(s - s_0)$ on Γ

$$\boxed{G = G(x, s)} \qquad \boxed{u = \int_\Gamma G(x,s) f(s)\, ds}$$



$T_t = T_{xx}$ $\quad x > 0 \quad t > 0$

$T = \sigma$ on $x = 0$

$T = f$ on $t = 0$

Solution will $\quad \boxed{T = T_1 + T_2}$

T_1 satisfies $(T_1)_t = (T_1)_{xx}$ $x>0,\ t>0$

$T_1 = 0$ on $x=0$

$= f$ on $t=0$

T_2 satisfies $(T_2)_t = (T_2)_{xx}$ $x>0,\ t>0$

$T_2 = \sigma$ on $x=0$

$T_2 = 0$ on $t=0$

Solve for T_1

$-T(-x,t)$ soln if $T(x,t)$ soln

T_1 problem is equivalent to

$$\begin{cases} T_t = T_{xx} & -\infty < x < \infty,\ t>0 \\ T = \bar{f}(x) \text{ for } t=0 \end{cases}$$

where $\bar{f}$ is odd and $\bar{f}(x) = f(x)$ ($x>0$)

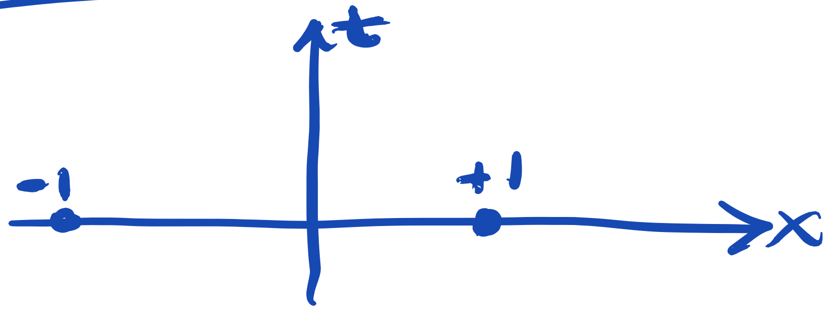
Green's function Solve with

i.d. $T = \delta(x-y) - \delta(x+y)$

$$\boxed{G(x,y) = \frac{1}{2\sqrt{\pi t}}\left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right]}$$

Green function for $\boxed{\begin{array}{ll} T_t = T_{xx} & x>0 \\ & t>0 \\ T = f & \text{at } t=0 \end{array}}$

$$\boxed{T = \int_0^\infty G(x,y)\, f(y)\, dy}$$



Same idea: $T_t = T_{xx}$ $\quad x>0,\ t>0$

$T = f \quad t=0$

insulated on $x=0$ $\longrightarrow$ $T_x = 0$ on $x=0$

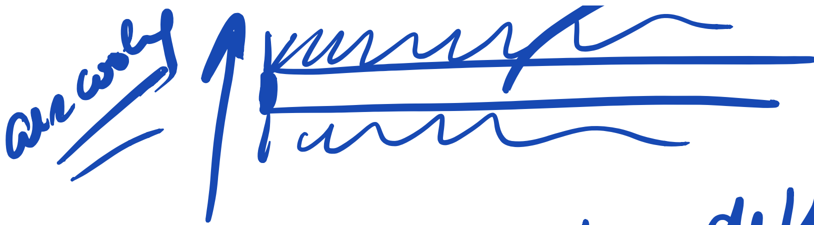
$$\boxed{G = \frac{1}{2\sqrt{\pi t}}\left[e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right]}$$

Third model Prescribe $T + \alpha T_x$ on $x=0$

$\Longrightarrow$ $T + \alpha T_x = 0$ on $x=0$

Robin B.C

$T_t = \nu T_{xx}$



heat flux $\propto$ Temperature difference

$$-\nu T_x = \mathbb{C}(T - T_0) \qquad \mathbb{C} > 0$$

$$\boxed{\nu T_x + \mathbb{C} T = \mathbb{C} T_0(t)}$$ Robin b.c.

$$T_x + \alpha T = 0 \qquad \alpha > 0$$

$$\boxed{T_t = T_{xx}} \longrightarrow T = G(x,y)$$

$$\delta(x-y)$$

$$\boxed{\phi = T_x + \alpha T} \qquad \boxed{\phi_t = \phi_{xx}} \quad x>0,\ t>0$$

$\phi = 0$ on $x=0$

and $\phi = \delta'(x-y) + \alpha\,\delta(x-y)$

Extend to $-\infty < x < \infty$ and solve with

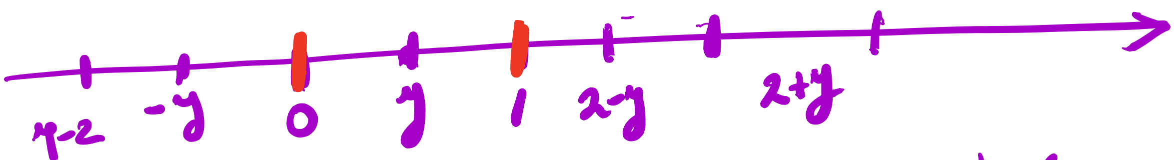
$$\boxed{\begin{aligned}\phi = {} & \delta'(x-y) + \alpha\,\delta(x-y) \\ & -\delta'(x+y) + \alpha\,\delta(x+y)\end{aligned}}$$

↑ Green's function ϕ.

$$T = e^{-\alpha x}\int^{x} e^{\alpha y}\phi(y)\,dy$$

Finite interval

$$T_t = T_{xx} \qquad 0<x<1$$
$$T_x = 0 \qquad \text{on } x=0,1$$
$$T = \delta(x-y) \qquad \text{for } t=0$$
some $0<y<1$



Deltas at $x = y + 2n$ | n integers

$-y + 2n$

$$G = \frac{1}{2\sqrt{\pi t}}\sum_n e^{-(x-y_n)^2/4t}$$

$\boxed{G = G(x,y,t)}$

$$T_t = T_{xx} \qquad 0<x<1$$
$$T = f(x) \qquad t=0$$
$$T = 0 \qquad \text{at } x=0,1$$

Can write soln two ways ✓

$$T = \int_0^1 G(x,y,t)\,f(y)\,dy$$

Good at t is small

What about the large

$$f(x) = \sum_1^\infty f_n \sin(nx)$$

$$T = \sum_1^\infty f_n \sin(nx) e^{-n^2 t}$$ ← Spectral method

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Eigenvalues and eigenvectors for self adjoint matrices

$\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$

$A = A^* \Rightarrow A$ has orthonormal basis of eigenvectors

Eigenvalues are real

$A\vec{u} = \lambda\vec{u}$

$$\langle A\vec{u}, \vec{u}\rangle = \langle \vec{u}, A\vec{u}\rangle = \bar{\lambda}\langle u, u\rangle$$

$A\vec{u} = \lambda u$, $\langle u, u\rangle = \|u\|^2$

$$\lambda \langle \vec{u}, \vec{u}\rangle \quad \|u\|^2 \qquad \lambda = \bar{\lambda}.$$

Take A has one eigenvector

$$A u_1 = \lambda_1 u_1 \qquad \|u_1\| = 1$$

Let H be set of all vectors that are $\perp$ u_1 — Hyperplane

Take $\vec{v} \in H$ then $A\vec{v}$ in H

$$\langle A\vec{v}, \vec{u}_1 \rangle = \langle \vec{v}, A\vec{u}_1 \rangle = \lambda_1 \langle \vec{v}, \vec{u}_1 \rangle = 0$$

$\therefore$ Can think of A as operator in H

Also self adjoint in H

Then use induction

For matrices this works also for normal matrices $\boxed{AA^{+} = A^{+}A}$

Again → has extension to ∞ dim.

If A is not normal big unknown for ∞ operators

Example self adjoint etc

Heat $T_t = \Delta u$ ← self-adjoint for reasonable B.C.

$$\langle u, \Delta v\rangle = \int_\Omega \underbrace{u\,\Delta v}_{\mathrm{div}(u\nabla v) - (\nabla u\cdot\nabla v)}\, dV =$$

$$= \int_{\partial\Omega} (u\nabla v)\cdot\hat{n}\, dA - \int_\Omega (\nabla u\cdot\nabla v)\, dV$$

$$\langle \Delta v, u\rangle = \int_{\partial\Omega} (v\nabla u)\cdot\hat{n}\, dA - \int_\Omega (\nabla u\cdot\nabla v)\, dV$$

if this vanishes then self-adjoint

$u_t = u_x + u_{xxx}$ $\qquad L = \partial_x + \partial_x^3$

skew-adjoint on $-\infty < x < \infty$

$$\langle Lu, v\rangle = -\langle u, Lv\rangle$$

$L^* = -L$

iL is self-adjoint

∴ eigenvalues of L purely im.

$u_t = \underbrace{u_x + u_{xxx}}_{Lu}$ | $x > 0$ plus some B.C. at $x=0$

$$\langle Lu, v\rangle = \int_0^\infty (u_x + u_{xxx})\, v\, dx =$$

$$= (uv)\Big|_0^\infty + (v u_{xx})\Big|_0^\infty - \int_0^\infty (uv + u_{xx}v)\, dx$$

$$= \underline{\underline{(uv + v u_{xx} - v_x u_x + v_{xx} u)\Big|_0^\infty}} - \underbrace{\int u\,(v_x + v_{xxx})\,dx}_{-\langle u, Lv\rangle}$$