Lecture 21 May 11, 2021 Tue

Green's function

Linear Algebra $Ax = y$ $\quad x = Gy$, $G = A^{-1}$

$x_n = \sum_m G_{nm} y_m$ componentwise

$AG = 1$ $\quad \underbrace{\sum_m A_{nm} G_{mk} = \delta_{nk}}_{\text{one system for every } k \quad k=1:N}$

Make it continuous

$Af = h$ $\quad G = G(x, y)$

A = linear operator on functions f

y fixed

$Ag = \delta(x-y)$

where $g(x) = G(x, y)$

G is the Green's function

Why important? ① Related to eigenvalues and spectrum

② Bdry / Boundary Int. methods for solving pde

③ Another way to write sln.

① Relationship with spectrum

For a matrix A, matrix valued function

$\underbrace{R(z) = (z-A)^{-1}}_{\text{Defined for any complex } z \text{ not an eigenvalue}}$

Resolvent { Analytic with poles at eigenvalues ($z=\lambda$)

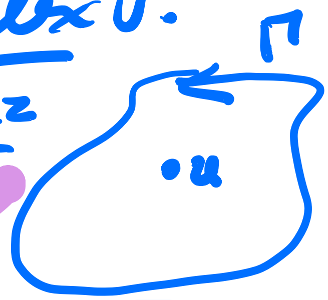
Note for operator

$R(z) = (z-A)^{-1}$ also resolvent

Why look at Green's function (or Resolvent) to get spectrum and not look at eigenvalues of discretized problem i.e. A matrix ≈ operator find eigenvalues of A

Cauchy's theorem in Complex V.

$$f(u) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\,dz}{z-u}$$



Γ enclosing u simple closed curve

Extension to matrices

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)\underbrace{(z-A)^{-1}}_{R(z)}\,dz$$

Say f = polynomial

need Γ to enclose all singul. of $R(z)$!

Now take $f = 1$

$$\boxed{I = \frac{1}{2\pi i}\int_\Gamma R(z)\,dz}$$

Resolution of the identity

Take any vector v

$$v = \frac{1}{2\pi i}\int_\Gamma R(z)\cdot v\, dz$$

do by residues $\longrightarrow$ write v in terms of eigenvectors of A

works for any A

Also extends to operators

Some caveats

"Best" way to implement this is Laplace Transform

Green's function B.V.P. for o.d.e.

Simple Case

$$\boxed{u'' - u = f(x)} \quad -\infty < x < \infty$$

"A" $= \partial_x^2 - 1$

$$\boxed{G_{xx} - G = \delta(x-y)}$$

$\mathcal{L}$ $u = \int_{-\infty}^{\infty} G(x,y) f(y)\, dy$

$\mathcal{L}u = \int_{-\infty}^{\infty} (\mathcal{L}G) f(y)\, dy = \int_{-\infty}^{\infty} \delta(x-y) f(y)\, dy = f(x)$

Find G. for $x \neq y$ $G_{xx} - G = 0$

$\therefore G \propto e^{-x}, e^{x}$

For $x > y$ $\quad G(x,y) = a(y) e^{-x}$

For $x < y$ $\quad G(x,y) = b(y) e^{x}$

at y need (i) G continuous

and (ii) $G_x(y+0, y) = G_x(y-0, y) + 1$

(i) $\Rightarrow$ $\boxed{a(y) e^{-y} = b(y) e^{y}}$

(ii) $\boxed{-a(y) e^{-y} = b(y) e^{y} + 1}$

$a(y) = b(y) e^{2y}$

$-b(y) e^{y} = b(y) e^{y} + 1$

$2 b e^{y} = 1$

$b = \frac{1}{2} e^{-y}$ $a = \frac{1}{2} e^{y}$

For $x > y$ $G = \frac{1}{2} e^{y-x}$

For $x < y$ $G = \frac{1}{2} e^{x-y}$ $= \frac{1}{2} e^{-|x-y|}$

Green's function for

$u'' - u = f(x)$ $G = \frac{1}{2} e^{-|x-y|}$

$\therefore u = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\,dy$

$\boxed{u'' - u = f}$ $0 < x < 5$

$u = 0$ at $x = 0$

$u = 0$ at $x = 5$

$G(x,y) = \alpha(y) \sinh x$ for $x < y$

$G(x,y) = \beta(y) \sinh(x-5)$ for $x > y$

$$G(x,y) = \begin{cases} K \sinh x \sinh(y-5) & \text{for } x < y \\ K \sinh(x-5) \sinh y & x > y \end{cases}$$

Symmetry $\boxed{G(x,y) = G(y,x)}$

in the example above Why?

Reason is that $\mathcal{L}$ is symmetric

$\boxed{\mathcal{L} = \partial_x^2 - 1}$ $\int f(x)\,\mathcal{L}g = \int \mathcal{L}f\, g\, dx$

Symmetric means have a scalar product

and $\boxed{\langle u, \mathcal{L}v\rangle = \langle \mathcal{L}u, v\rangle}$

For matrices means $A = A^T$

real

For complex matrix $A = A^+$

transpose conjugate

$\langle u, Av\rangle = u^T(\overline{Av})$

$= (\overline{Av})^T u = \bar{v}^T A^+ u = \langle Au, v\rangle$

$Av_n = \lambda_n v_n$ $\quad \langle v_n, v_m\rangle = \delta_{nm}$

$G = \sum_n \frac{1}{\lambda_n} v_n v_n^T$ $\quad \lambda_n \neq 0$

$$G = \sum_n \frac{\quad}{\lambda_n}$$

$$(GA)v_m = \lambda_m G v_m = \lambda_m \sum_n \frac{1}{\lambda_n} v_n \underbrace{v_n^T \cdot v_m}_{\delta_{nm}}$$

$$= v_m$$

→ Go now To green's function

$$G = \sum_n \frac{1}{\lambda_n} \varphi_n(x)\varphi_n(y)$$

PDE Example #1

$$\boxed{\begin{gathered} T_t = \nu T_{xx} \\ T = f(x) \text{ at } t = 0 \\ -\infty < x < \infty \end{gathered}}$$

Find Green's Funct

Find

$$\boxed{\begin{gathered} G_t = \nu G_{xx} \quad \text{for } t > 0 \\ G = \delta(x-y) \quad \text{at } t = 0 \\ -\infty < x < \infty \quad \text{and} \; -\infty < y < \infty \end{gathered}}$$

#1 Can set $y = 0$ because of Translation Invariance ✓

ν dimensions of $\text{length}^2/\text{time}$

$$\xi = \frac{x}{\sqrt{\nu t}}$$ nondim quantity

$$G = * \, f(\xi) =$$

$$\int_{-\infty}^{\infty} G \, dx = q$$

dimension of G are $\frac{q}{x}$

$$G = q * f(\xi)$$

$$G = \frac{q}{\sqrt{\nu t}} f(\xi)$$

t

$$T_t = \nu T_{xx}, \quad T = \delta(x) \text{ at } t = 0$$