Lecture 20, Th May 6 2021

Finish hyperbolic part

Simple Wave

$\vec{u}_t + A(\vec{u})\vec{u}_x = 0$ $\quad$ A real diagonalizable

Eigenvalue $\quad \vec{l}_n A = \lambda_n \vec{l}_n$

$A\vec{r}_n = \lambda_n \vec{r}_n$

Look for the

$$\vec{u} = \vec{U}(\varphi) \qquad \boxed{\varphi = \varphi(x,t)}$$

Simple wave

Eqn becomes

$$\boxed{\left(\varphi_t + \varphi_x A(U)\right)\frac{d\vec{U}}{d\varphi} = 0}$$

$\therefore$ solution

$$\boxed{\frac{d\vec{U}}{d\varphi} = \vec{r}\,(\vec{U})}$$

and

$$\boxed{\varphi_t + \lambda(\vec{U})\,\varphi_x = 0}$$

Scalar

Develops multiple values and breaks $\quad \boxed{\dfrac{d\lambda}{d\varphi} \neq 0}$

$$\frac{d\lambda}{d\varphi} = (\nabla_{\vec{u}}\lambda)\cdot\vec{r} = (\vec{r}\cdot\nabla_u)\lambda \neq 0$$

coefficient of "genuine nonlinearity"

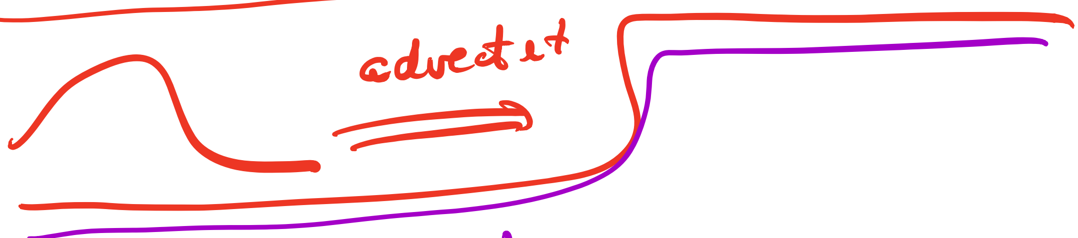
#1] $(\vec{r}_u \cdot \nabla_u)\lambda \neq 0$ $\quad$ Eigenvalue is genuinely nonlinear

#2] $(\vec{r}_u \cdot \nabla_u)\lambda \equiv 0$ $\quad$ Linearly degenerate

Example Gasdynamics

Sound Waves are genuinely nonlinear

Particle paths are linearly degenerate



General Outlook

#1) To solve $U_t + A(u)u_x = 0$

need shocks & conservation form

and "proper" physics

i.e. produce shocks

i.e. neglected effects are dominated by dissipation

What do shock look like?

Cons. Form $\quad \vec{u}_t + (\vec{F}(u))_x = 0$

Rankine Hugoniot

s be shock speed $\quad \boxed{-s[u]+[F]=0}$

$\vec{u} \simeq$ nice part $+ H(x-st)\vec{v}$

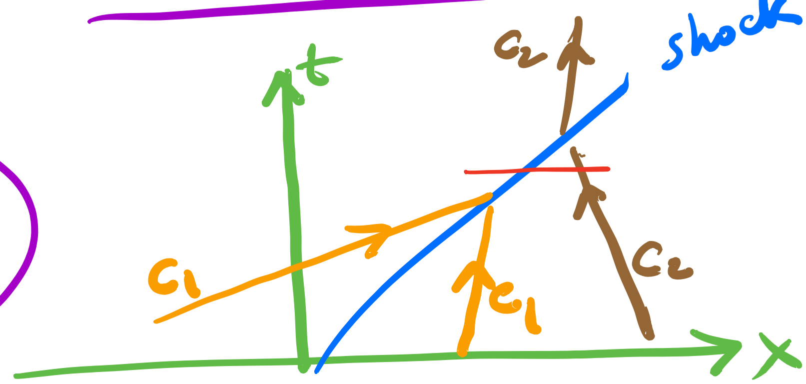
Well produce several branches of these say "n of them"

example in Gas dynamics get 3 !!

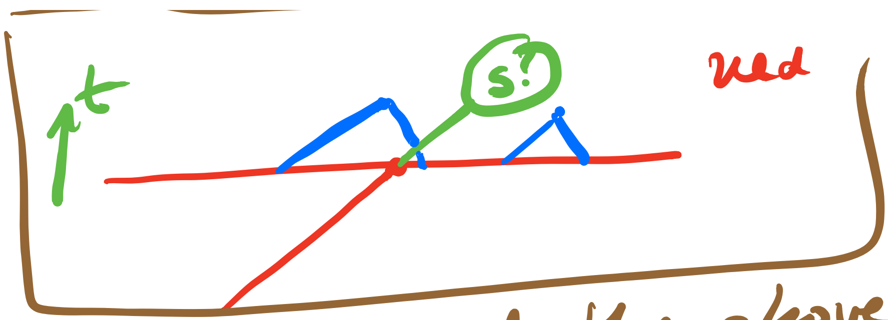
Need entropy conditions too !!

[because vs complicated

Simple example 2x2 system



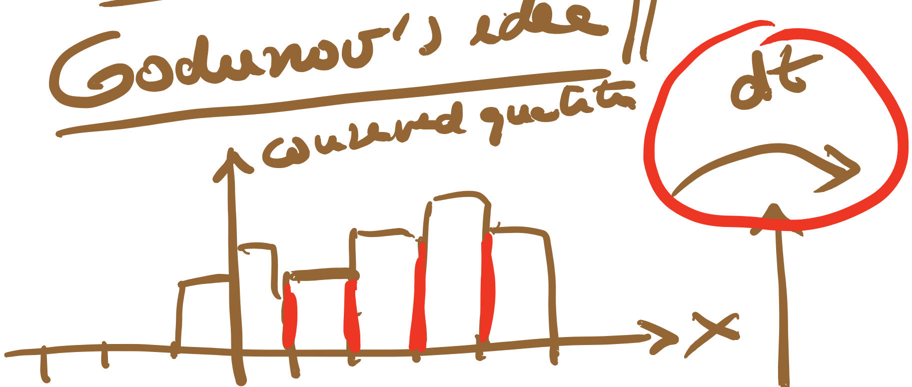
Lax entropy condition



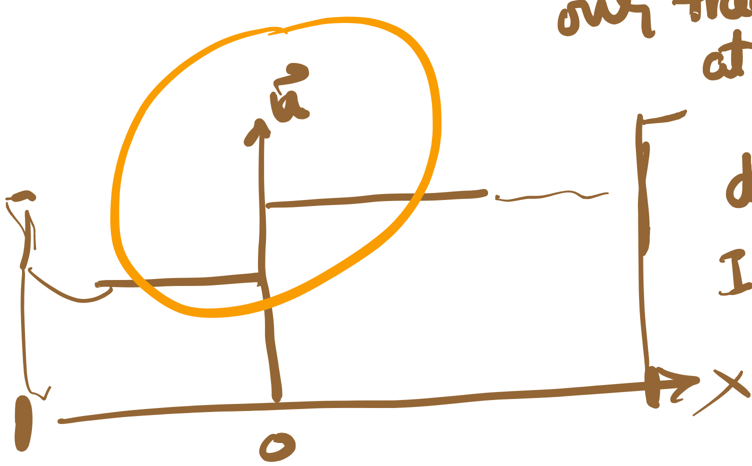
Given that all the above works

How do I solve the equation ??

Godunov's idea ||



only thing I need is fluxes!
at edges.

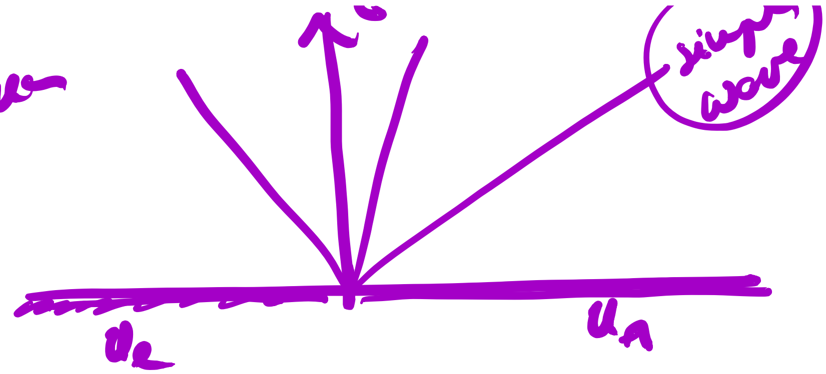


dt small enough
I can ignore
the other
interfaces

Riemann Problem

Piecewise
initial data

Typical Sln. Riemann Prob



Extension to n-d ??

$$\vec u_t + \mathrm{div}[\vec F(\vec u)] = 0$$

or

$$\underbrace{\vec u_t + (F_1)_{x_1} + (F_2)_{x_2} + \dots + (F_n)_{x_n} = 0}$$

$$\rho_t + \mathrm{div}(\rho \vec u) = 0$$

$$(\rho \vec u)_t + \mathrm{div}[\rho \vec u \otimes \vec u + I p] = 0$$

$$\mathrm{div}[T_{ij}] = \sum \frac{\partial}{\partial x_i} T_{ij}$$

$\boxed{\text{What does it mean to be hyperbolic?}}$ in this case

It means it is hyperbolic in every direction!!

It means $\bar{u}_t + (n \cdot F)_{\tilde{x}} = 0$

is hyperbolic for any $\hat{n}$ unit vector

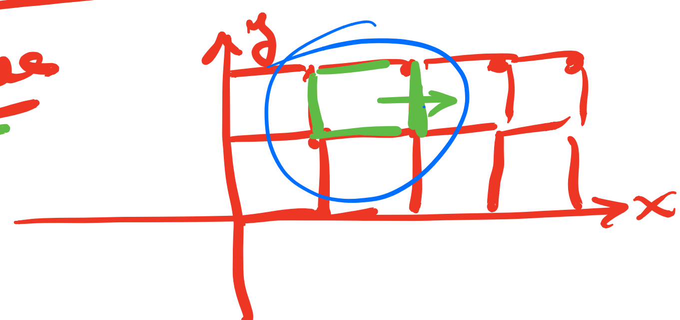
$$(F_j)_{x_j} = A_j(\vec{u})\,\vec{u}_{x_j}$$

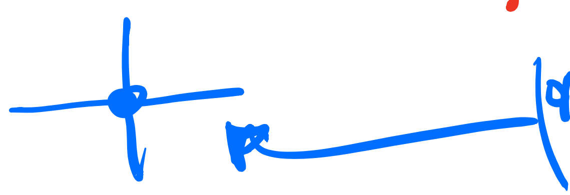
hyperbolic means $\sum \alpha_j A_j = \mathcal{A}$

real diagonalizable for any choice of constants α_j

Next how do I solve it??

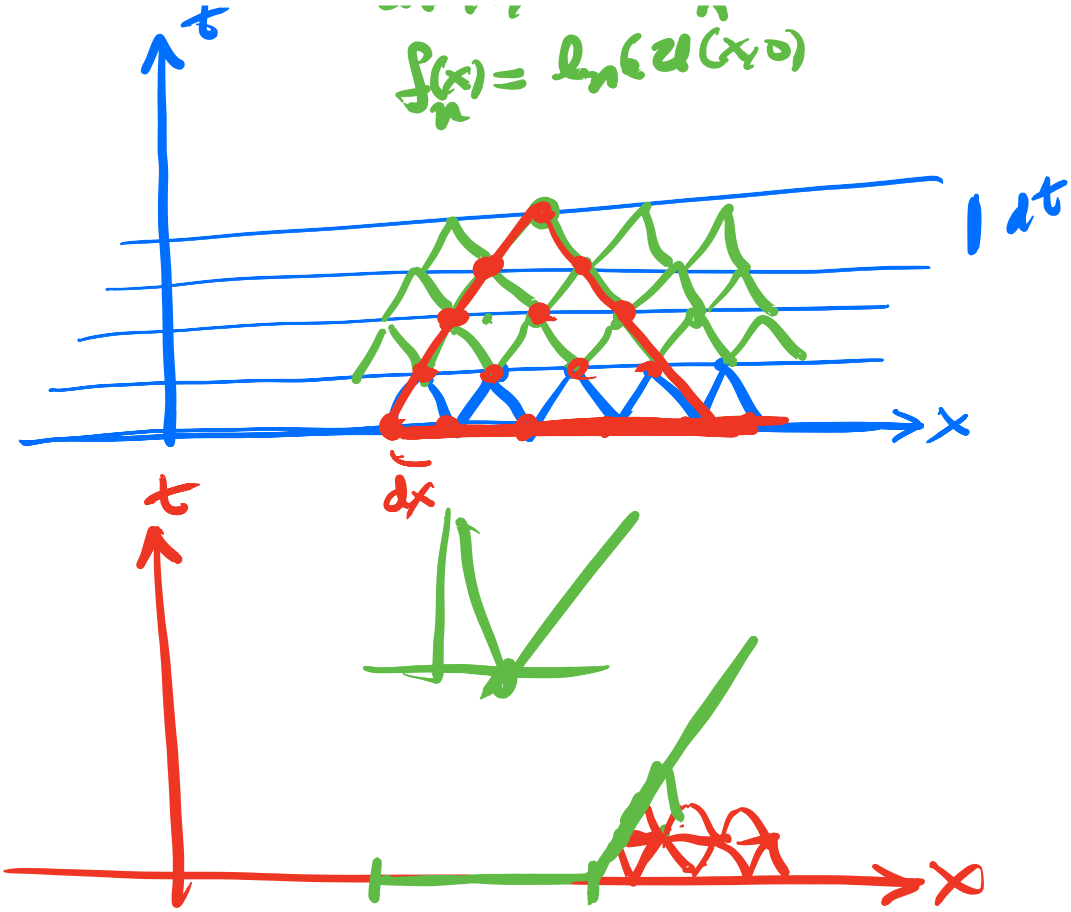
"Godunov" idea





$$\boxed{u = \sum f(x - \lambda_n t)\, r_n}$$

$u(x,0) = \sum f_n(x)\, r_n$



$f_n(x) = \ln(u(x,0))$
t
x
dt
dx
t
x