Lecture 18

$$\underbrace{a u_{xx} + b u_{xy} + c u_{yy} + L.O.T. = 0}$$

[A] $a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2 = 0$ $\quad \boxed{\phi = C}$

Curve where eqn. can have singularities of the "weak" type

Define symbol of eqn $\boxed{p = ak_1^2 + bk_1k_2 + ck_2^2}$

1) Hyperbolic p can be factorized as product of 2 real polynomials (distinct)
2) Elliptic $p = 0$ only trivial solution
3) Parabolic p is square of real polynomial

1 means that [A] $\Longleftrightarrow$ $\alpha\phi_x + \beta\phi_y = 0$ or $\gamma\phi_x + \delta\phi_y = 0$

Can write $p = \vec{k}\cdot\begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}\vec{k} = \vec{k}\cdot A\vec{k}$

and A is symmetric $\therefore$ ~ diagonal matrix

$\therefore$ change of coordinates

$p = \lambda_1 k_1^2 + \lambda_2 k_2^2$ or in terms of ϕ $\quad \boxed{0 = \lambda_1\phi_{\eta_1}^2 + \lambda_2\phi_{\eta_2}^2}$

Hyperbolic means λ_1 & λ_2 are nonzero and opposite sign. $\lambda_1 = \alpha^2$, $\lambda_2 = -\beta^2$

$$0 = \alpha^2 \phi_{\eta_1}^2 - \beta^2 \phi_{\eta_2}^2 = (\alpha \phi_{\eta_1} + \beta \phi_{\eta_2})(\alpha \phi_{\eta_1} - \beta \phi_{\eta_2})$$

Elliptic means λ_1 & λ_2 are nonzero and both have the same sign

Parabolic means one of the λs vanishes

$$\boxed{a u_{xx} + \text{L.O.T.} = 0}$$ modulo rotations

Why the name

Hyperbolic $\quad p(k_1, k_2) = \mathbb{C} = \lambda_1 k_1^2 + \lambda_2 k_2^2$

$\leftarrow$ hyperbolas!

Elliptic $\quad p(k_1, k_2) = \mathbb{C} = \lambda_1 k_1^2 + \lambda_2 k_2^2$

Ellipses

Parabolic case $\quad x^2 = \mathbb{C}$

$\hookrightarrow \quad x^2 + c_1 x + c_2 y = \mathbb{C}$

parabolas $\longrightarrow$

Example degenerate dominant term

Suppose try to solve $\qquad 0<\epsilon \ll 1$

$$f(x) + \epsilon g(x) + O(\epsilon^2) = 0$$

If $f(x_*) = 0$ then 9 has soln nearby which is $O(\epsilon)$ near it.

↑ simple root

double root splits and nearby ~~splits one~~ $O(\sqrt{\epsilon})$ near!

parabolic eqn $\sim$ $\boxed{u_{xx} + \alpha u_y = 0}$

First order "systems" of eqns [quasi-linear]

$$\boxed{\vec{u}_t + A\vec{u}_x = \vec{F}}$$

$A = A(x, \vec{u}, t)$

$F = F(x, \vec{u}, t)$

Example Gas Dynamics

$$\boxed{\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + u u_x + \frac{1}{\rho} p_x = 0 \end{aligned}}$$

$$\boxed{p = p(\rho)}$$

Isentropic

Case 1 "simplest" $\quad$ A constant

$\vec{u} = \vec{u}(\phi, \psi)$ "sep. along $\phi = \text{const}$"

$$\boxed{(\phi_t + \phi_x A)\vec{u}_\phi + (\psi_t + \psi_x A)\vec{u}_\psi = F}$$

$u_t = \phi_t u_\phi + \psi_t u_\psi$
$u_x = \phi_x u_\phi + \psi_x u_\psi$

Sln. where $\vec{u}_\phi$ is discontinuous
↳ nontrivial $\iff$ $[\vec{u}_\phi]$ has to be an eigenvector
and $\det[\phi_t + A\phi_x] = 0$
i.e. $\phi_t = \lambda \phi_x$ where
$\lambda =$ eigenvalue of A

---

at $\phi = 0$ u_ϕ is discontinuous
on $\phi = 0+$ $[\phi_t + \phi_x A]\, u_\phi^+ + \text{rest} = 0$
on $\phi = 0-$ $[\phi_t + \phi_x A]\, u_\phi^- + \text{rest} = 0$

$$[\phi_t + \phi_x A]\,[u_\phi] = 0$$

$\phi_t = -\lambda \phi_x$ $\Rightarrow$ $\phi_x[-\lambda + A][u_\phi] = 0$

Condition get "characteristics"

$\Longleftrightarrow$ A has real eigenvalues

Definition: If A is real diagonalizable

Hyperbolic

Elliptic: All eigenvalues of A are non-real

Examine characteristics

λ_n is an eigenvalue of A with eigenvector v_n

Ch. Solve $\phi_t = \lambda \phi_x$ & find $\phi = 0$

equivalent curve $\boxed{\frac{dx}{dt} = \lambda}$

$\boxed{\vec{u}_t + A\vec{u}_x = F}$

$\frac{dx}{dt} = \lambda_n$, λ_n eigenvalue of A

characteristics

Simple Case suppose A is const, F=0

v_n = eigenvectors

$$\vec{u} = \sum \sigma_n v_n$$

$$\sum \left[(\sigma_n)_t + \lambda_n (\sigma_n)_x \right] v_n = 0 \quad \text{Eqn}$$

$\therefore \quad (\sigma_n)_t + \lambda_n (\sigma_n)_x = 0$

$\therefore$ general solution $\boxed{\vec{u} = \sum \sigma_n (x - \lambda_n t) \vec{v}_n}$

Complicate things a bit: A not const.

but A still real diagonalizable.

$A \vec{v}_n = \lambda_n \vec{v}_n \quad \Big| \quad n = 1 \ldots N = \mathrm{size}(A)$

v_n are linearly indep.

λ_n are all real

Linear algebra can look at left Eigenvectors

$$\vec{\ell}_n \cdot A = \lambda_n \vec{\ell}_n$$

& can normalize so $\vec{\ell}_n \cdot \vec{v}_m = \delta_{nm}$

$\vec{u}_t + A \vec{u}_x = \vec{F}$

$\lambda_n = \lambda_n (x, t, u)$

etc

left multiply by ℓ_n

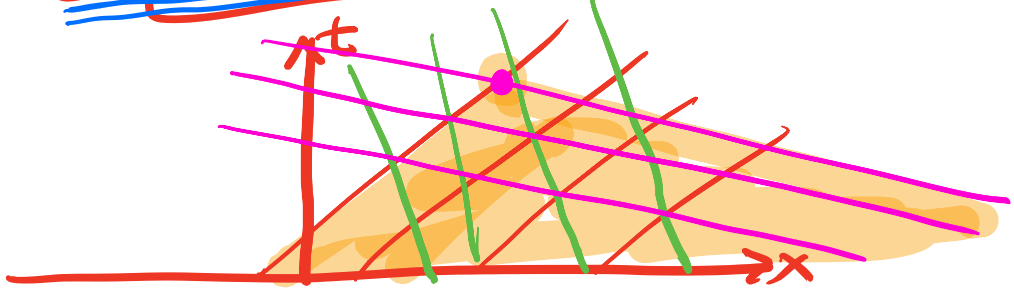
$$\boxed{\vec{l}_n \cdot [\vec{u}_t + \lambda_n \vec{u}_x] = l_n \cdot F \qquad n = 1 \dots N}$$

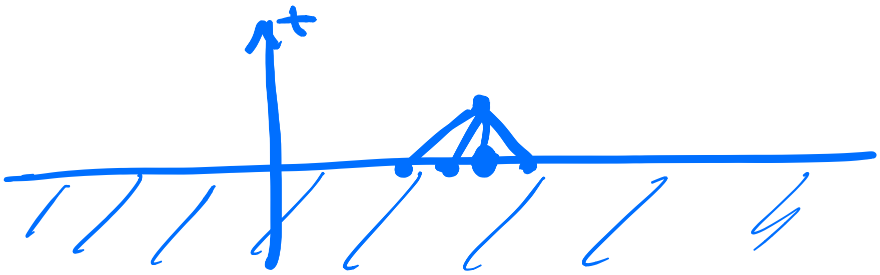
Equivalent to original eqn

Introduce Charact

$$\boxed{\frac{dx}{dt} = \lambda_n}$$

$$\boxed{\vec{l}_n \cdot \frac{d\vec{u}}{dt} = \vec{l}_n \cdot \vec{F} \quad \text{along}}$$





$t_0 \to dt$