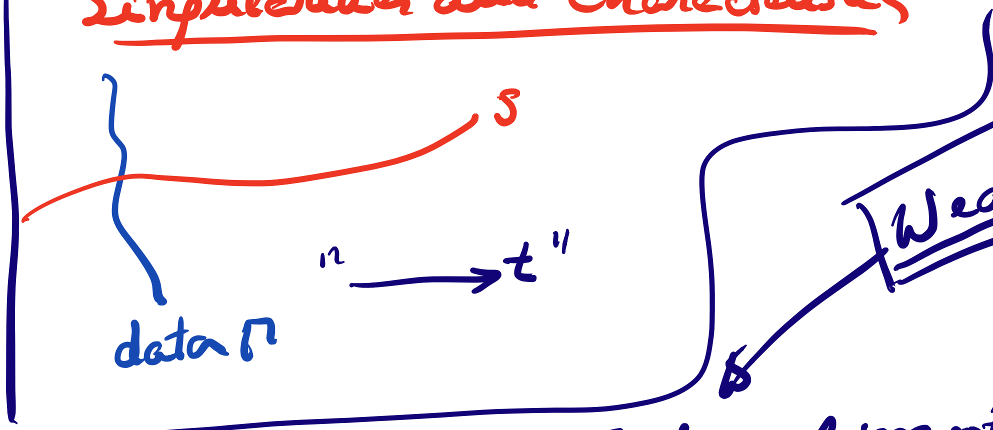
Singularities and Characteristics



Weak lines.

Def. of "singularity": Failure of smoothness that still allows pde to make sense.

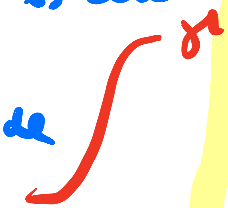
Study next: "strongest" "weak singularity" — SWS

Example $\boxed{a u_x + b u_y = c}$ (1)

$a = a(x,y,u)$
$b = b(x,y,u)$
$c = c(x,y,u)$

SWS means u is cont.

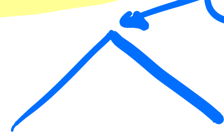
There is a curve γ; u is C^1 on each side of γ and derivatives have limit on curve as approached, but they may be different on each side.



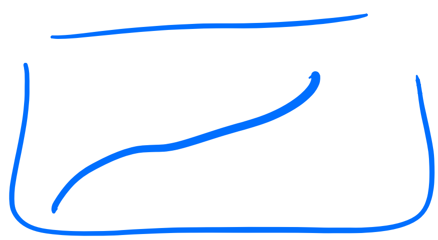
Example

Mountains

SWS

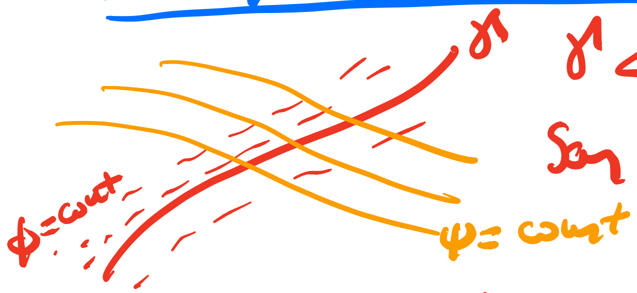


Γ = water divide line



Does (1) allow this?

Realize curve as a level set

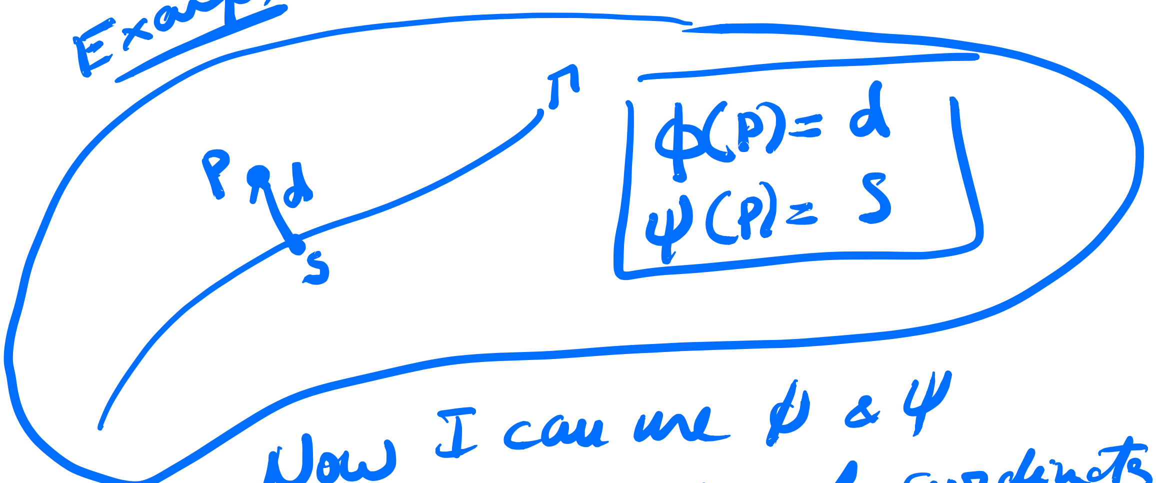


$\gamma \Longleftrightarrow \phi(x,y) = 0$

Say ϕ = signed distance to γ

Let $\psi(x,y)$ be another function such that $\nabla\psi$ not aligned with $\nabla\phi$

Example



Now I can use ϕ & ψ as a new system of coordinates!

$$\Rightarrow \quad \begin{aligned} a &= a(\phi,\psi,u) \\ b &= b(\phi,\psi,u) \\ c &= c(\phi,\psi,u) \end{aligned} \qquad \begin{aligned} u_x &= \phi_x u_\phi + \psi_x u_\psi \\ u_y &= \phi_y u_\phi + \psi_y u_\psi \end{aligned}$$

$$\boxed{(a\phi_x + b\phi_y)u_\phi + (a\psi_x + b\psi_y)u_\psi = c}$$

Now: SWS means u_ϕ jumps across $\phi=0$

but u_ψ is continuous

Then are all continuous

$$\Rightarrow \boxed{a\phi_x + b\phi_y = 0}$$

curve with SWS exist and satisfy

Say curve γ is $x = x(s)$, $y = y(s)$ then

$$\phi(x(s), y(s)) = 0$$

$$\boxed{\phi_x \dot{x} + \phi_y \dot{y} = 0} \quad (\dot{x}, \dot{y}) \perp \nabla\phi$$

$$(a,b) \perp \nabla\phi$$

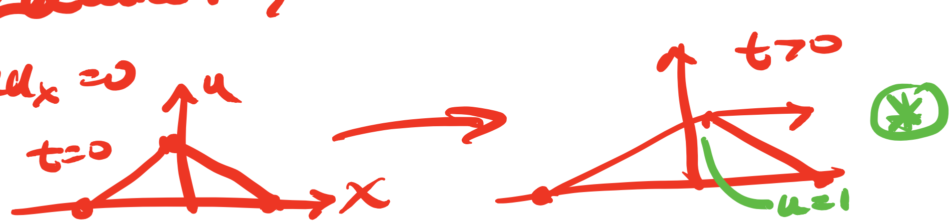
$\therefore (\dot{x}, \dot{y}) \parallel (a,b)$

$$\boxed{\frac{dx}{ds} = \alpha a, \quad \frac{dy}{ds} = \alpha b}$$

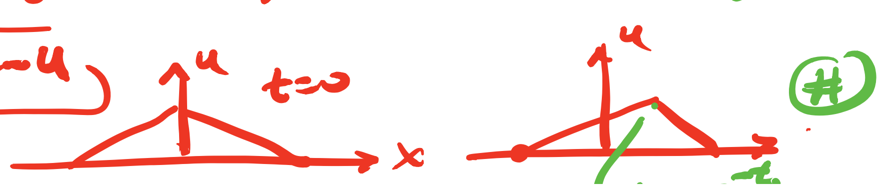
Ch. Eq.

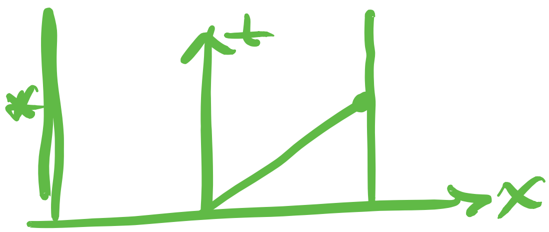
Example "Lecture N, N<17"

$u_t + uu_x = 0$



$\boxed{u_t + uu_x = -u}$





Move out of scalar 1st order

Check weak singularities for 2nd order eqn

$$\boxed{a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f = 0} \quad (2)$$

$a = a(x, y, u, u_x, u_y)$, same for the rest

Can (2) have SWS?? i.e. derivatives exist up to second order, but second der. are discontinuous across some curve

Again introduce coordinate system $\boxed{\phi, \psi}$ so that $\phi = 0$ is the "curve"

Means: $u, u_\phi, u_\psi, u_{\psi\psi}$ and $u_{\psi\phi}$ are all cont. but $u_{\phi\phi}$ may jump plug into eqn. (2)

$a \to a(\phi, \psi, u, u_\phi, u_\psi)$ "nice" cont.

(Same for b, c, d, e, f)

No problem with u_x, u_y

$u_{xx} = (\phi_x u_\phi + \psi_x u_\psi)_x =$

$= \phi_x^2 u_{\phi\phi} + \text{cont. terms}$ $\therefore$ eqn. $\Longrightarrow$

$u_{yy} = \phi_y^2 u_{\phi\phi} + \text{cont. terms.}$

$$(a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2)u_{\phi\phi} + \text{cont. stuff} = 0$$

Conclusion

$$\boxed{0 = a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2} \quad (3)$$

as eqn. that curve must satisfy $(\phi = 0)$

$\therefore$ (2) allows characteristics iff (3) has solutions

Example "Wave eqn" $c^2 u_{xx} - u_{yy} = 0$

$c^2\phi_x^2 - \phi_y^2 = 0 \quad \therefore (c\phi_x = \pm\phi_y)$

$$c\phi_x = -\phi_y \Rightarrow \boxed{\frac{dx}{dy} = c}$$

$$c\phi_x = \phi_y \Rightarrow \boxed{\frac{dx}{dy} = -c}$$

$\phi(x,y)=0$ $\quad \phi_x\,dx + \phi_y\,dy = 0$ | $c\,dy = dx$

$$c\phi_x + \phi_y = 0$$

Example Laplace $u_{xx} + u_{yy} = 0$

$$\Longrightarrow \phi_x^2 + \phi_y^2 = 0$$ NO nontrivial sln.

Laplace eqn has no charact
does not allow w. singularities

General Case $\quad a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2 = 0$

Case 1 can factor polynomial

$$ak_1^2 + bk_1k_2 + ck_2^2 = S(\vec{k})$$

$$= (\alpha k_1 + \beta k_2)(\gamma k_1 + \delta k_2)$$

real

Then get "curves" $\alpha \phi_x + \beta \phi_y = 0$

or $\gamma \phi_x + \delta \phi_y = 0$

$\therefore$ have charact.

In this case Equ. is hyperbolic definition

Case 2 Cannot factor $S(\vec{k})$ with real factors

Then equ. is definition Elliptic

Case 3 "limit" case.

Can factor $S(k)$ but 2 factors are equal.

only one set of ch. (but double)

parabolic (example heat equ

$u_t = u_{xx} \Longrightarrow \boxed{\phi_x^2 = 0}$ S

$$S(\vec{k}) = ak_1^2 + bk_1k_2 + ck_2^2$$

Symbol of the pde

$$\boxed{u_{tt} + x\, u_{xx} = 0}$$

$\underline{x < 0}$ hyperbolic and $c = \sqrt{-x}$

$\boxed{\underline{x > 0} \text{ is elliptic}}$

$\boxed{\text{Tricomi Equ.}}$