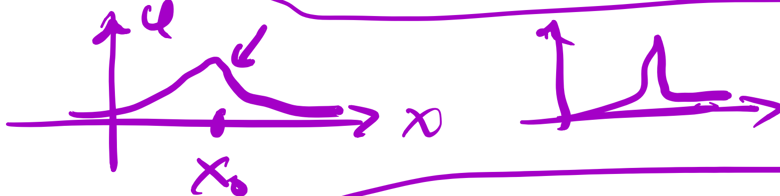
Lecture 16

Function $f = f(x)$, $-\infty < x < \infty$

$$L_f \varphi = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

φ = test functions

Set of φ = all smooth (∞ derivative) functions that "vanish" rapidly at ∞ i.e. φ and all derivatives vanish



$$\int_{-\infty}^{\infty} f'(x)\varphi(x) = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx$$

$$\boxed{L_{f'}(\varphi) = -L_f(\varphi')} \; *$$

Example take f = Heaviside function $= \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x>0 \end{cases}$

$$L_{H'}(\varphi) = -\int_{-\infty}^{\infty} H(x)\,\varphi'(x)\,dx$$

$$= -\int_0^{\infty} \varphi'(x)\,dx = \varphi(0)$$

$$\text{"="} \quad \boxed{\int_{-\infty}^{\infty} H'(x)\,\varphi(x)\,dx}$$

$$\underline{\underline{H'(x) = \delta(x)}}$$

What about f'' ??

$$\boxed{L_{f''}(\varphi) = -L_{f'}(\varphi') = L_f(\varphi'')}$$

$$\int_{-\infty}^{\infty} H''(x)\,\varphi(x)\,dx = \int_{-\infty}^{\infty} H\,\varphi''(x)\,dx$$

$$= \int_0^{\infty} \varphi''(x)\,dx = -\varphi'(0)$$

$$H'' = (\delta)' =$$

Notion $\boxed{\text{Generalized derivative}}$

Defined not just for functions but for "operators"

$$\underline{\underline{L : \{\text{set of } \varphi\text{'s}\} \longrightarrow R}}$$

"Distributions"

Example $f(x) = \log|x|$

$$L_f = \int_{-\infty}^{\infty} (\log|x|)\varphi(x)\,dx$$

$$L_{f'}(\varphi) = -\int_{-\infty}^{\infty} (\log|x|)\varphi'(x)\,dx$$

$$\text{"="} \int_{-\infty}^{\infty} \underbrace{(\log|x|)'}_{\frac{1}{x}}\varphi(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x}\varphi(x)\,dx = -\int_{-\infty}^{\infty} (\log|x|)\,\varphi'(x)\,dx$$

Def.

Turns out

$$= \text{principal value} =$$

$$= \lim_{\epsilon \downarrow 0} \int_{-\infty}^{-\epsilon} \frac{1}{x}\varphi(x)\,dx + \int_{\epsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x}\varphi(x)\,dx = \int_{-\infty}^{-1} \frac{1}{x}\varphi(x)\,dx + \int_{-1}^{1} \frac{\varphi(x)}{x}\,dx$$

$$+ \int_{1}^{\infty} \frac{1}{x}\varphi(x)\,dx = -\int_{-\infty}^{-1} \ln|x|\,\varphi'(x)\,dx$$

$$- \int_{1}^{\infty} \ln|x|\,\varphi'(x)\,dx$$

$$+ \int_{-1}^{1} \frac{\varphi(x)-\varphi(0)}{x}\,dx + \cancel{\int_{-1}^{1} \frac{1}{x}\varphi(0)\,dx}$$

$$= -\int_{-1}^{1} (\ln|x|)\,[\varphi(x)-\varphi(0)]'\,dx$$

$$= -\int_{-\infty}^{\infty} (\ln|x|)\, \varphi'(x)\, dx$$

Formally

$$\int_{-\infty}^{\infty} \frac{1}{x^2} \varphi(x)\, dx = \int_{-\infty}^{\infty} (\ln|x|)\, \varphi''(x)\, dx$$

$$\uparrow \; (\ln|x|)''$$

Poisson eqn $\boxed{\Delta \Phi = f}$

$$\boxed{\Delta G = \delta(\vec{x} - \vec{y})} \qquad G = G(\vec{x}, \vec{y})$$

$$\underbrace{\Phi = \int G(\vec{x}, \vec{y})\, f(\vec{y})\, dy_1 \cdots dy_n}$$

G = Green's function

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$$

"$\Delta \phi = \delta(x)$" means what

$$\iint (\Delta\Phi)\,\varphi \,dx\,dy = \iint \Phi\, \Delta\varphi \,dx\,dy$$

$\phi_{xx} + \phi_{yy}$

What is a δ function in 2-D

in 2-D $\quad \boxed{\delta = \frac{1}{2\pi} \Delta \ln |\vec{r}|}$

Application to Cons. Laws 1-D

$$\boxed{\rho_t + q_x = f}$$

Take test function

$\varphi = \varphi(x,t)$ $\quad \varphi$: defined $-\infty < x < \infty,\ t > 0$

φ vanishes for $|x| \to \infty$ or $t \to \infty$

$$\int_0^\infty dt \int_{-\infty}^\infty dx\, f\varphi = \int_{-\infty}^\infty dx \int_0^\infty dt\, \rho_t \varphi + \int_0^\infty dt \int_{-\infty}^\infty q_x \varphi \,dx$$

$$= -\int_{-\infty}^{\infty} dx \int_0^{\infty} dt \,(\rho \varphi_t + q \varphi_x) +$$

$$-\int_{-\infty}^{\infty} \rho(x,0)\,\varphi_t \, dx$$

True for all φ's

$$\boxed{\int_{-\infty}^{\infty} dx \int_0^{\infty} dt \left[f\varphi + \rho \varphi_t + q \varphi_x \right] + \int_{-\infty}^{\infty} \rho(x,0)\,\varphi_t \, dx = 0}$$

F. Elements — First take test function

finite $\boxed{\varphi = \sum a_n \varphi_n}$

$\{\varphi_n\}$ "basis" for test functions

Soln $\boxed{\rho = \sum \rho_n \phi_n}$

Connection with Fourier

$$f(x) = \sum_{-\infty}^{\infty} f_n e^{inx}$$

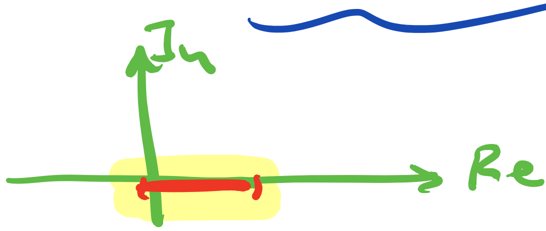
plug into Test function technology

↑ smooth periodic functions

$$\int f(x)\varphi(x)\,dx$$

$$= \sum \int_0^{2\pi} f_n \varphi(x) e^{inx}\,dx = 2\pi \sum f_n \overline{\varphi_n}$$

where $\varphi_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x) e^{-inx}\,dx$



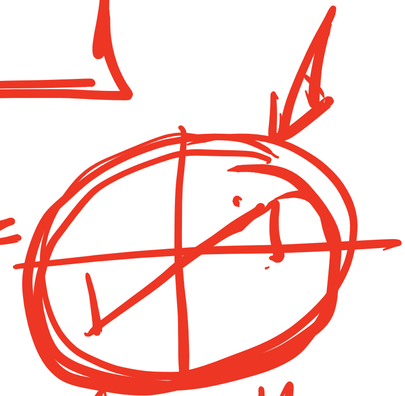
$$f(x) = \sum_{-\infty}^{\infty} \frac{1}{(1+n^2)^2} e^{inx} \quad ✓$$

$$f'(x) = \sum_{-\infty}^{\infty} \frac{in}{(1+n^2)^2} e^{inx} \quad ✓$$

$$f^{IV}(x) = \sum_{-\infty}^{\infty} \frac{n^4}{(1+n^2)^2} e^{inx} \quad ?$$

$$\boxed{f(x)=\sum_{-\infty}^{\infty} c_n e^{inx}}$$

$$f(x)=\sum_{\substack{-\infty \\ n\neq 0}}^{\infty}\frac{(i)^n}{n}e^{inx} =$$



sawtooth

$= x$ for $|x|<\pi$

periodic

period 2π

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{-inx}\,dx}$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-inx}}{in}\,dx + x\frac{e^{-inx}}{in}\Big|_{-\pi}^{\pi}$$

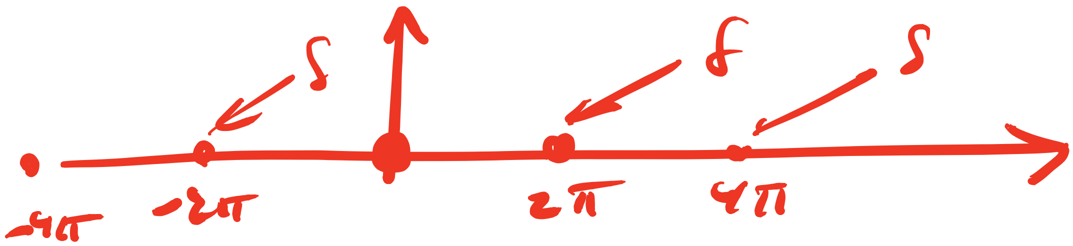
$$= \frac{2}{in}$$

What is the Fourier series for the δ function?

Specifically $f(x)=\sum_{-\infty}^{\infty}\delta(x-2n\pi)$

$-\infty$

array of δ's at integers $\times 2\pi$



$$a_n = \int_{-\pi}^{\pi} f(x)\, e^{-inx} \frac{dx}{2\pi} = \frac{1}{2\pi}$$

$$\boxed{f(x) = \frac{1}{2\pi} \sum e^{inx}}$$

What is the Fourier Transform of $\delta(x)$??

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = \frac{1}{2\pi}$$

<u>Derivative</u> $f'(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} in\, e^{inx}$