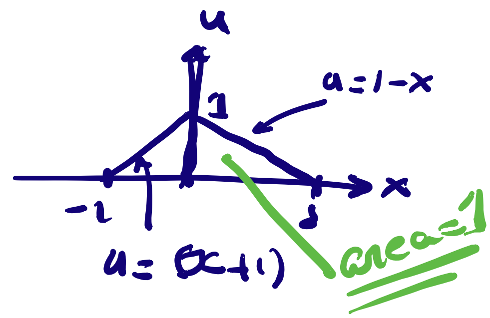
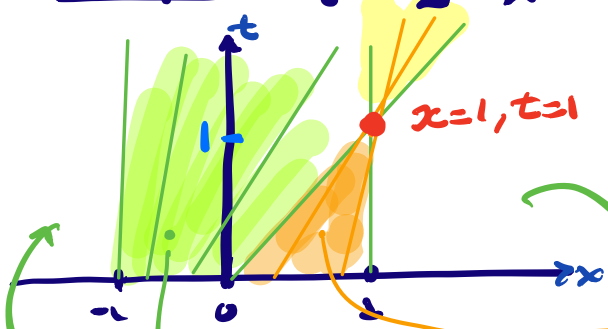
Lecture 15 2021/04/15

Example $u_t + \left(\frac{1}{2}u^2\right)_x = 0$





$u = 0$, $x = \xi$

$u = 1 - \xi$
$x = (1-\xi)t + \xi$

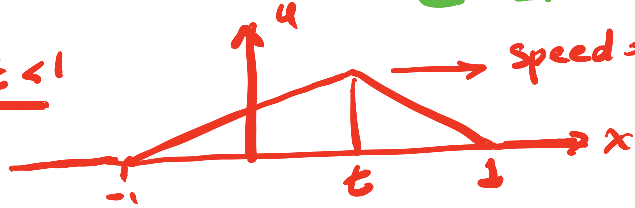
$u = \dfrac{-\,x-t}{1-t}$

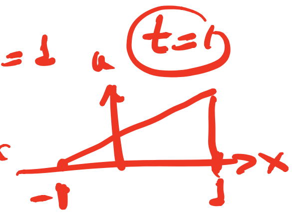
$u = 0$
$x = \xi$

$u = 1 + \xi$
$x = (1+\xi)t + \xi$

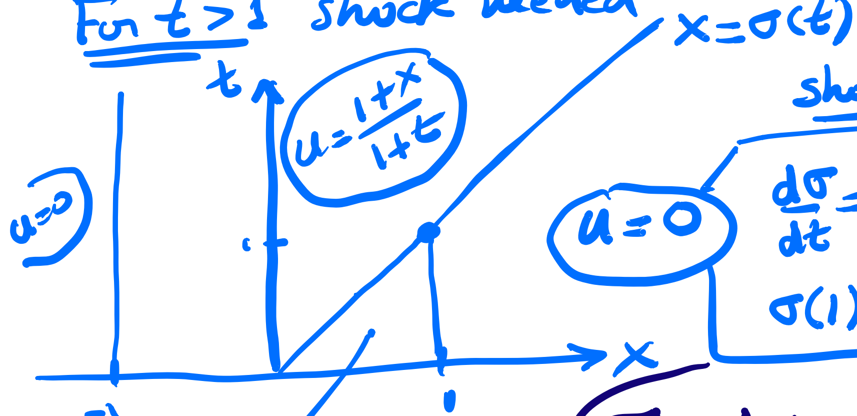
$u = 1 + \dfrac{x-t}{1+t}$

For $0 < t < 1$





For $t > 1$ shock needed



shock speed

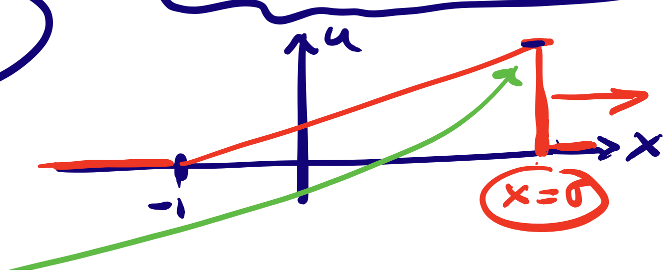
$u = 0$

$$\frac{d\sigma}{dt} = \frac{1}{2}\,\frac{1+\sigma}{1+t}$$

$$\sigma(1) = 1$$

$$\boxed{\sigma = -1 + \sqrt{2(1+t)}}$$

For $t > 1$



$$u = \frac{\sqrt{2(1+t)}}{1+t} = \sqrt{\frac{2}{1+t}}$$

Triangle Base length $= \sqrt{2(1+t)}$

height $= \sqrt{\frac{2}{1+t}}$

$\therefore$ area $= 1$

Area $= 1$

"Variation" Suppose we solve

$$\boxed{u_t + \left(\tfrac{1}{2}u^2\right)_x = -au}$$

same initial data

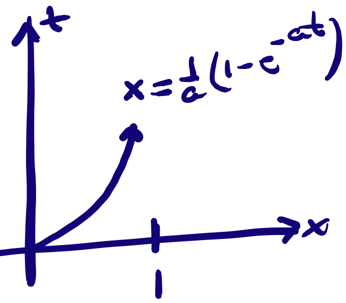
sln. by ch.

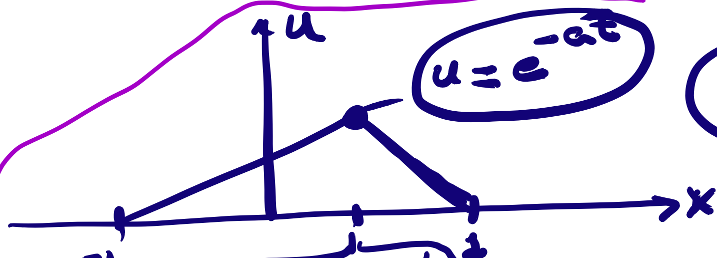
$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = -au$$

$a > 0$

$$u = u_0(\xi)\, e^{-at}$$

$$x = \frac{1}{a} u_0(\xi)\left(1 - e^{-at}\right) + \xi$$





$u = e^{-at}$

$x = \frac{1}{a}(1 - e^{-at})$

If $a > 1$

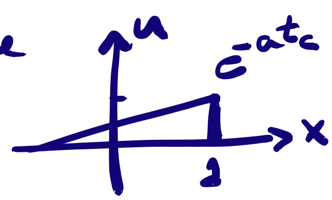
valid $0 < t < \infty$

no shock

If $a < 1$

Then critical time

$$\boxed{1 = \frac{1}{a}\left(1 - e^{-at_c}\right)}$$



Area e^{-at}

$\boxed{u_t + (\tfrac{1}{2}u^2)_x = -au}$ Shock conditions

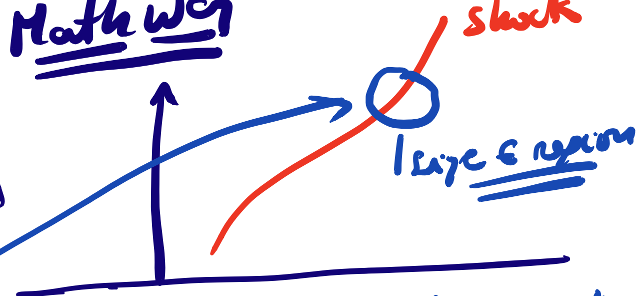
Same as $u_t + (\tfrac{1}{2}u^2)_x = 0$

i.e. Ignore source term

$$\frac{d}{dt}\int_a^b u\,dx = \frac{1}{2}u^2\Big|_a - \frac{1}{2}u^2\Big|_b + \int_a^b u\,dx$$

Split ↑ $a < x < \sigma$ and $\sigma < x < b$

Math way

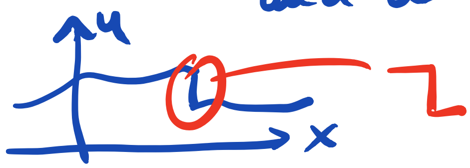


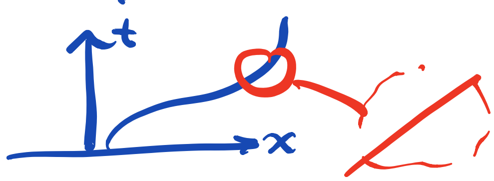
$$\boxed{\begin{array}{c} u_t + q_x = f(u) \\ q = q(u) \end{array}}$$

Stretch region by $\left(\frac{1}{\epsilon}\right)$ to make "size 1"

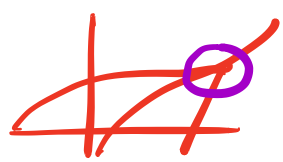
equ becomes $\boxed{\frac{1}{\epsilon}[u_{\tilde{t}} + q_{\tilde{x}}] = f}$

+ shock curve becomes straight
and u " piecewise const.





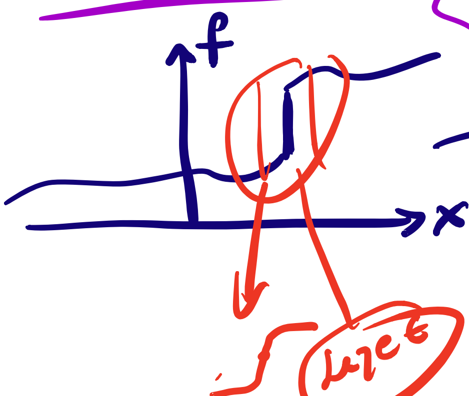
$c = \frac{dq}{du}$ $\quad \frac{dx}{dt} = c(u)$ & $\frac{du}{dt} = f$



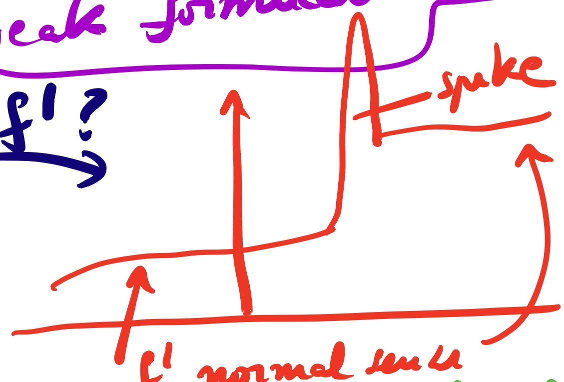
Argument "Singular stuff" in pde controlled by highest derivative

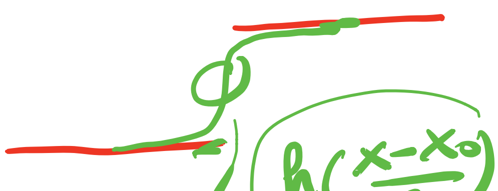
"Approach 1"

Weak derivative and weak formulation



f'?

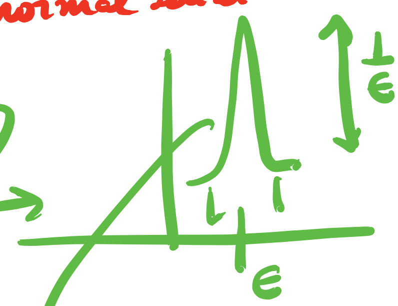




$h\left(\frac{x-x_0}{\epsilon}\right)$

$\int_a^b f(x)\,dx$

spike



property it has

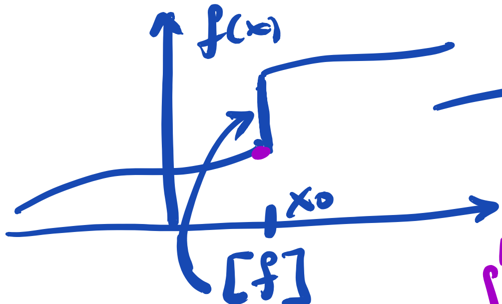
Isolate spike and place it at $x=0$

$\longrightarrow$ lim "F"

$F=0$ outside $x=0$

$\int_a^b F(x)\,dx = 1$ if $a<0<b$

Dirac's delta function $\delta(x)$



$$f' = f' + [f]\,\delta(x-x_0)$$

"normal"

$$\int_a^b f'\,dx = f(b) - f(a) \quad \text{"true"}$$

If $[a,b]$ does not include x_0, "obvious".

If $a < x_0 < b$ then:

$$\int_a^b f'\,dx = \int_a^{x_0-dx} f'\,dx + \int_{x_0-dx}^{x_0+dx} f'\,dx + \int_{x_0+dx}^{b} f'\,dx$$

$$= \cancel{f_\ell} - f(a) + (\cancel{f_r} - \cancel{f_\ell}) + f(b) - \cancel{f_r}$$

Use this to get Rankine – Hugoniot

$$\boxed{\rho_t + q_x = s}$$

ρ discontinuous along $x = \sigma(t)$

$\therefore$ "locally" $\rho = \rho(x - \sigma(t))$

$$\rho_t \to \rho_t\,\text{"normal"} + (-\dot{\sigma})[\rho]\,\delta(x-\sigma)$$

$$q_x \to q_x\,\text{"normal"} + [q]\,\delta(x-\sigma)$$

To satisfy eqn $\boxed{p_t + q_x = S \text{ away from } x = \sigma}$
need
$+ \quad \underline{-\dot{\sigma}[p] + [q] = 0}$ R.H. cond.

$\boxed{\text{Proper theory}}$ Formally, for any function $\varphi = \varphi(x)$

$$\boxed{\int_a^b \delta(x)\varphi(x)\,dx = \varphi(0)}$$

δ only makes sense when inside an integral