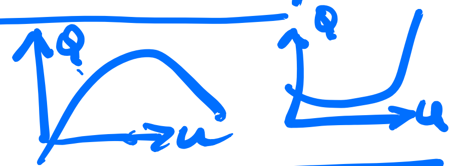
Lecture 14, 2021/04/13 [Example]

"Shock Structure"

$$\boxed{u_t + \varphi(u)_x = \epsilon(\nu u_x)_x}$$

φ is convex or concave | $\varphi''(u) \neq 0$

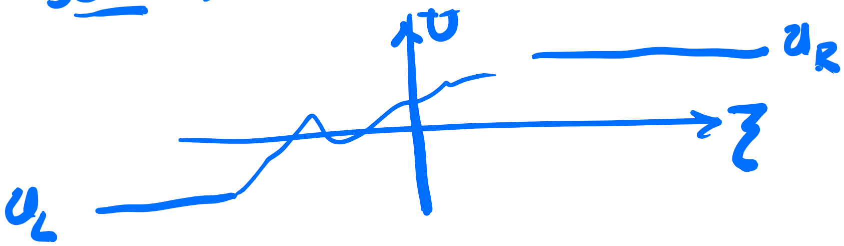


$\nu = \nu(u) > 0$

$u = U\left(\frac{x-st}{\epsilon}\right)$ $\xi = \frac{x-st}{\epsilon}$

$-sU' + \varphi(U)' = (\nu U')'$ ode

Solve: assume $U \to$ const as $\xi \to \pm\infty$



$u_R = \lim_{\xi \to \infty} U$

$u_L = \lim_{\xi \to -\infty} U$

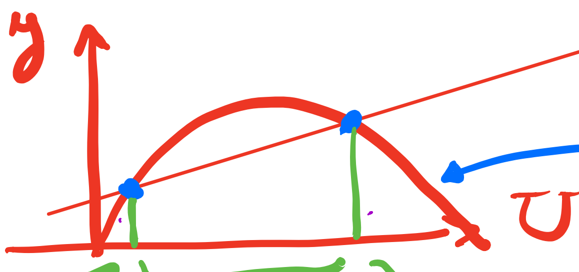
$\nu U' = \varphi - sU + K$

$$\boxed{U' = \frac{1}{\nu}\left[\varphi - sU + K\right]}$$

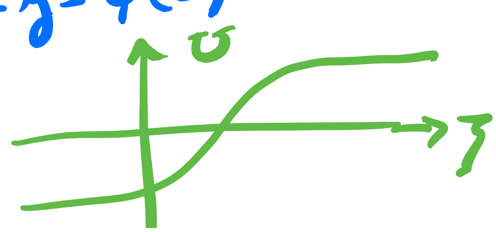
diff between

$y = sU - K$

$y = \varphi(U)$



u_L $U' > 0$ (Sln)



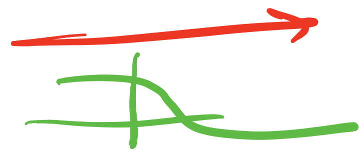
u_L $\quad$ u_R $\quad$ is $O(\epsilon)$ in x

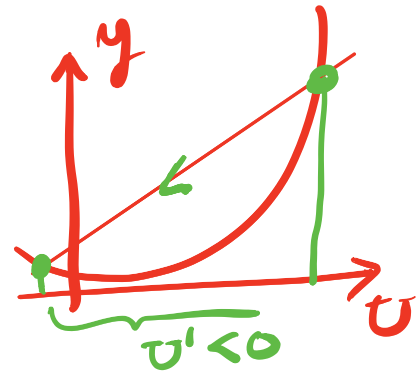
Graphically clear

(a) Rankine Hugoniot satisfied

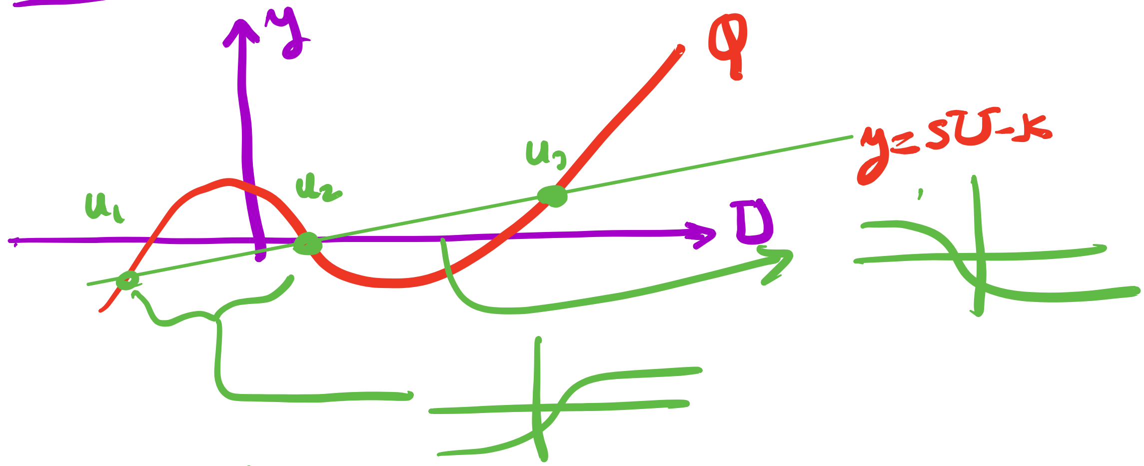
(b) Lax entropy satisfied!

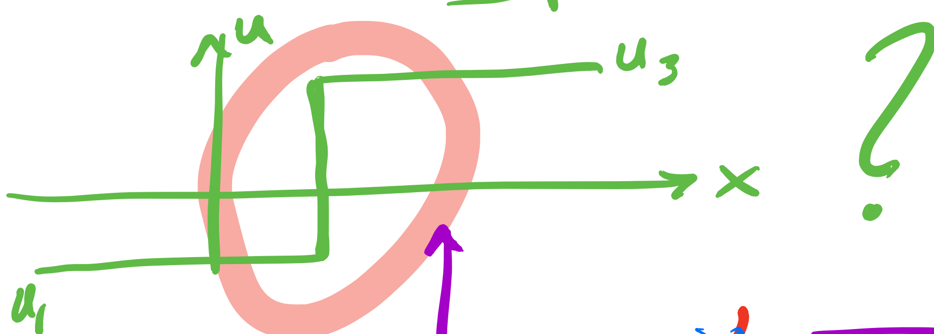
Φ convex

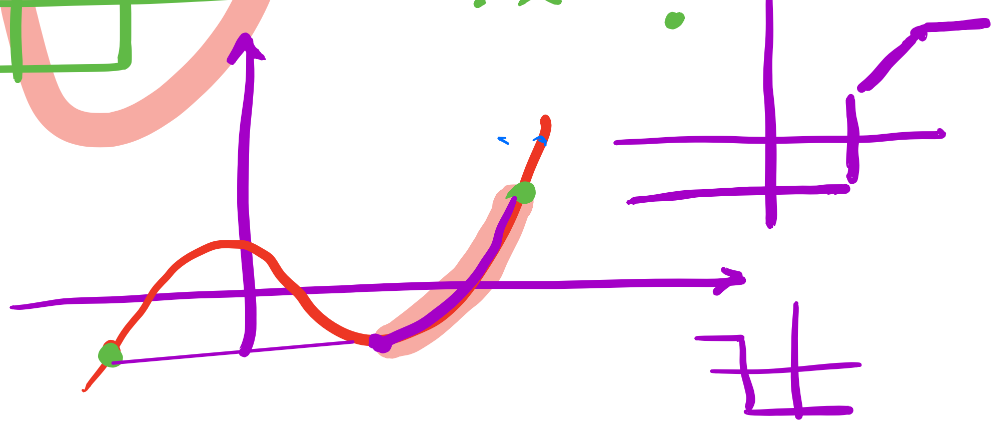


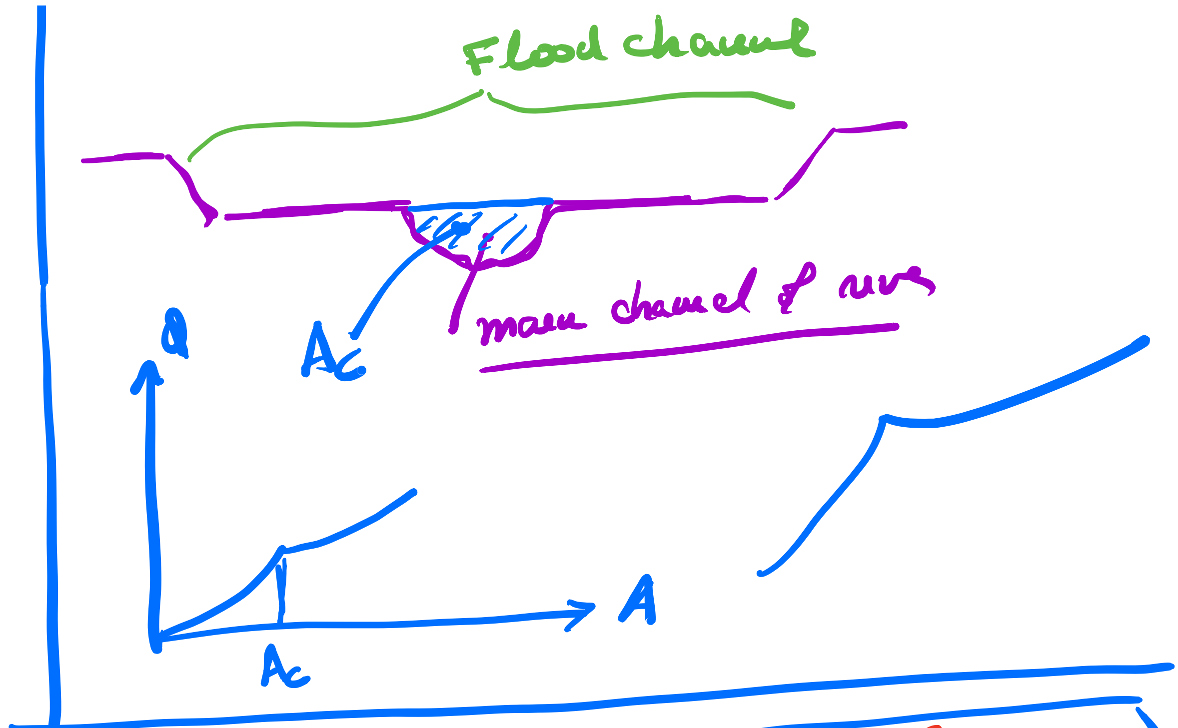


Sidebar: What if Φ is not concave nor convex?









"Energy/Entropy"

$$\boxed{u_t + \Phi_x = \epsilon(\nu u_x)_x}$$

multiply by u

$$u u_t + \underbrace{u(\Phi(u))_x}_{u\,c(u)\,u_x} = \epsilon\, u(\nu u_x)_x$$

$$\boxed{\frac{df}{du} = u\,c(u)}$$

$$\left(\tfrac{1}{2}u^2\right)_t + (f)_x = \epsilon\left[(\nu u u_x)_x - \nu u_x^2\right]$$

$$\left(\tfrac{1}{2}u^2\right) + \left[f - \epsilon \nu u u_x\right]_x = -\epsilon \nu u_x^2$$

$$\frac{d}{dt}\int_a^b \left(\tfrac{1}{2}u^2\right)dx = \left[f - \epsilon \nu u u_x\right]_a^b \overbrace{- \epsilon\int_a^b \nu u_x^2\,dx}^{\text{dissipation}}$$

$$\boxed{u = U\left(\frac{x-st}{\epsilon}\right)}$$

and let $a \to -\infty,\ b \to \infty$ **Negative**

$$\frac{d}{dt}\int_A^B \left(\tfrac{1}{2}u^2\right)dx = \left[f - \nu U U'\right]\Big|_A^B - \int_A^B \nu (U')^2\, d\xi$$

Dissipation in shock region

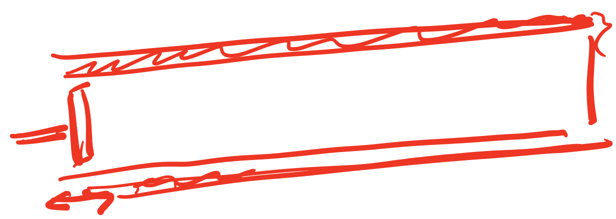
Width of region is $O(\epsilon)$

Derivatives are $O(\epsilon^{-1})$

Dissipation $=$ width $\times\, \epsilon \times$ 2 derivatives $= O(1)$

viscous coeff.

illustration



$$\boxed{u_t + \left(\tfrac{1}{2}u^2\right)_x = 0}$$

example

- u conserved
- Flux is $\frac{1}{2}u^2$
- shocks allowed

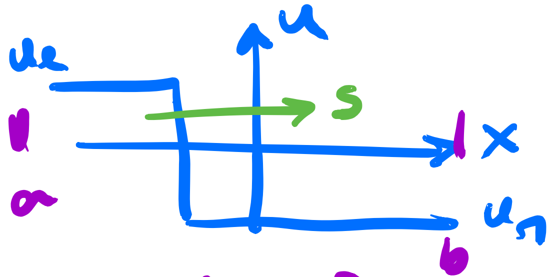
(1) charact. speed is u

(2) Flux is convex $\frac{d^2 q}{du^2} = 1 > 0$ shock decrease u

③ R. Hugoniot

$$s = \frac{[q]}{[u]} = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r} = \frac{1}{2}(u_r + u_\ell)$$

shock speed = average of ch. speeds !



Can look at $\int_a^b \frac{1}{2}u^2\,dx$ where shock is in between a and b

σ = shock position

$$\frac{d}{dt}\int_a^b \frac{1}{2}u^2\,dx = \frac{d}{dt}\left(\int_a^\sigma \frac{1}{2}u^2\,dx + \int_\sigma^b \frac{1}{2}u^2\,dx\right) =$$

$$= \frac{d}{dt}\left[\frac{\sigma - a}{2}u_\ell^2 + \frac{b-\sigma}{2}u_r^2\right] =$$

$$= \frac{1}{4}(u_r + u_\ell)u_\ell^2 - \frac{1}{4}(u_r + u_\ell)u_r^2 =$$

$$= \left(\frac{1}{3}u_\ell^3 - \frac{1}{3}u_r^3\right)$$

$$+ \left[\frac{1}{4}u_r u_\ell^2 + \underbrace{\cancel{\frac{1}{4}u_\ell^3}}_{-\frac{1}{12}u_\ell^3} - \frac{1}{4}u_r^3 - \underbrace{\cancel{\frac{1}{4}u_\ell u_r^2}}_{+\frac{1}{12}u_r^3}\right]$$

$$= \frac{1}{3}(u_\ell^3 - u_r^3)\ \boxed{-\frac{1}{12}\left[u_\ell^3 - 3u_\ell^2 u_r + 3u_\ell u_r^2 - u_r^3\right]}$$

$$\boxed{\begin{gathered} u_t + u u_x = 0 \\ \Updownarrow \cdot u \\ \left(\tfrac{1}{2}u^2\right)_t + \left(\tfrac{1}{3}u^3\right)_x = 0 \\ \text{if no shocks} \end{gathered}}$$

$= \frac{1}{12}(u_\ell - u_r)^3$ positive!

shock diss.