Lecture 13, 2021 April 8, Th

$$\boxed{c = \frac{dQ}{d\rho}} \qquad \Big/ \qquad \rho_t + q_x = 0$$

$$q = Q(\rho)$$

$$\frac{dc}{d\rho} \neq 0$$

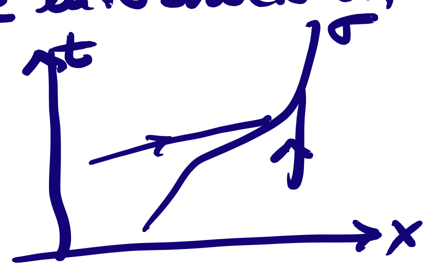
Slns

ρ = piecewise smooth with discontinuities along $x = \sigma(t)$ shocks

Shocks satisfy $\frac{d\sigma}{dt} = \frac{[q]}{[\rho]}$ Rankine Hugoniot

$\rightarrow$ $c_{left} > \frac{d\sigma}{dt} > c_{right}$ Entropy

Characteristics converge into shock on each side



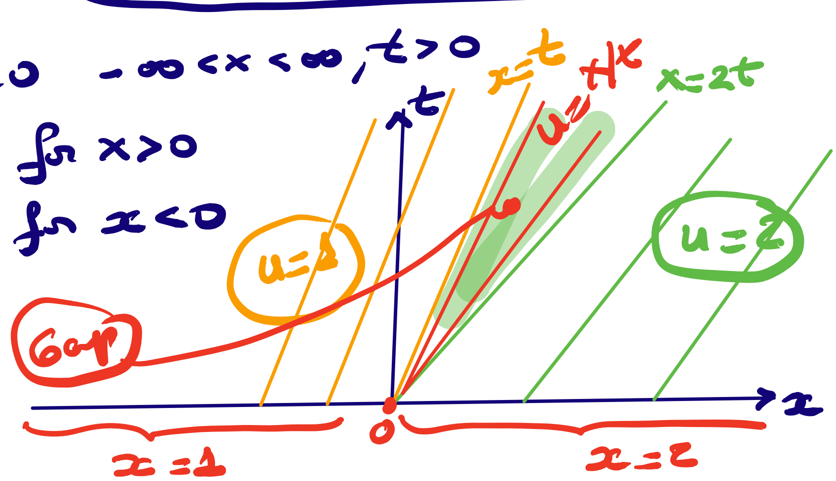
Example

$u_t + uu_x = 0 \qquad -\infty < x < \infty,\ t > 0$

$u(x,0) = 2$ for $x > 0$

$u(x,0) = 1$ for $x < 0$

$\boxed{\text{ch. speed} = u}$



smear

at $x=0, t=0$ initial data takes all the values $1 \le u \le 2$

$\therefore$ Ch. $x = ut$ $\quad 1 \le u \le 2$

$\therefore u = \frac{x}{t}$

Expansion Fan fills gap

Expansion fan works if at discontinuity with ρ_1 on the left, ρ_2 on the right

and $c(\rho_1) < c(\rho_2)$

also need $c(\rho)$ monotone between ρ_1 and ρ_2 so $c(\rho) = \frac{x}{t}$ can be inverted

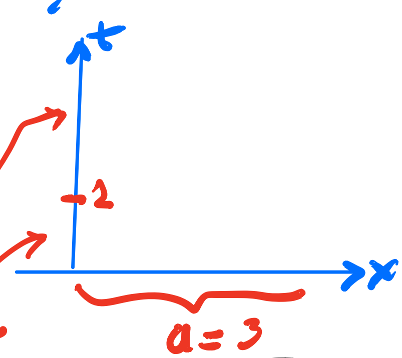
Example 2

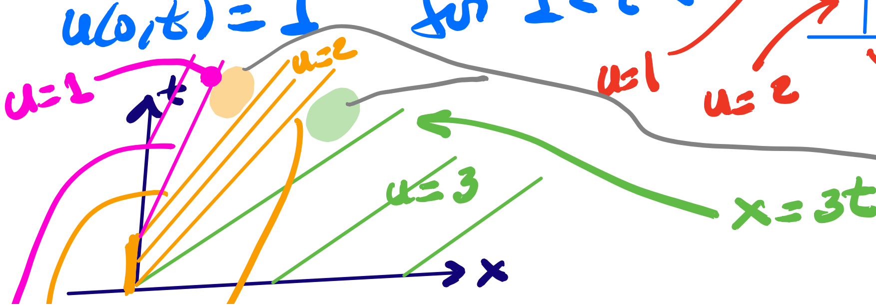
$u_t + u u_x = 0 \qquad 0 < x < \infty, \; t > 0$

$u(x,0) = 3$ for $x > 0$

$u(0,t) = 2$ for $0 < t < 1$

$u(0,t) = 1$ for $1 < t < \infty$





$x=2t$ $\quad x=2(t-1)$ $\quad x=(t-1)$

Gap $2t < x < 3t$ Fill in with ch. that start at $x=0, t=0$, with all $2<u<3$

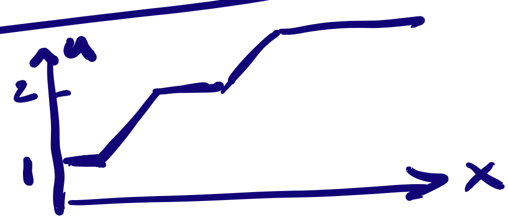
$x = ut$ $\quad \therefore \quad u = \frac{x}{t}$

Gap $(t-1) < x < 2(t-1)$ Fill in with ch. start at $x=0, t=1$ with all $1<u<2$

$x = u(t-1)$ $\quad u = \frac{x}{t-1}$

Full solution

$$u = \begin{cases} 1 & \text{for } 0 < x < t-1 \\ \frac{x}{t-1} & \text{for } t-1 < x < 2(t-1) \\ 2 & \text{for } 2(t-1) < x < 2t \\ \frac{x}{t} & \text{for } 2t < x < 3t \\ 3 & \text{for } x > 3t \end{cases}$$



Question why is u "linear" in the fans?

Reason fans arise from solving

$$c(p) = \frac{x - x_0}{t - t_0} \text{ in general}$$

where (x_0, t_0) is where the data has a jump

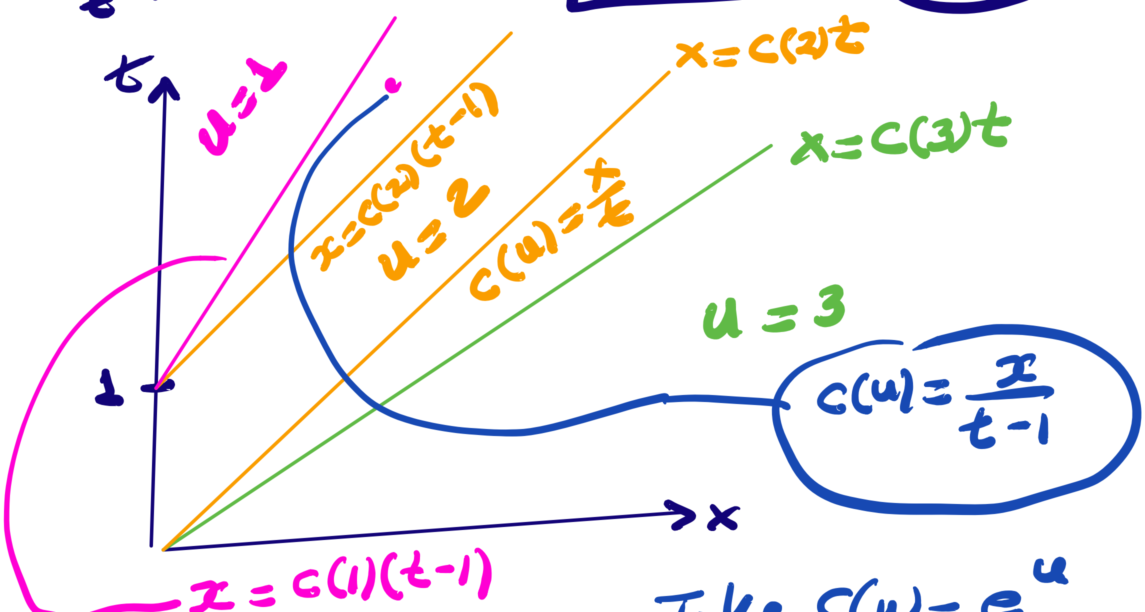
Now for $u_t + u u_x = 0$ $\boxed{c = u}$

$$u = \frac{x - x_0}{t - t_0}$$

but if $c(u)$ is not linear then different answer

Example 3 Same problem but

$u_t + c(u) u_x = 0$ $\boxed{\frac{dc}{du} > 0}$ same data



Take $c(u) = e^u$

in gap!

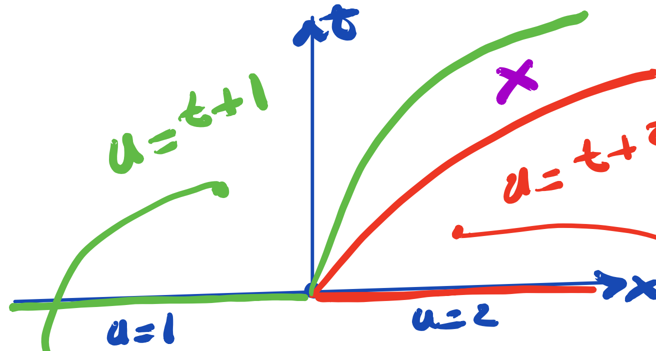
$u = \log\left(\frac{x}{t}\right)$ or $u = \log\left(\frac{x}{t-1}\right)$

Example 4

$u_t + uu_x = 1$

Ch. forms

$\frac{dx}{dt} = u, \quad \frac{du}{dt} = 1$



Ch. that starts at $x = \xi$, $t = 0$ with $\xi > 0$

$u = 2 + t$

$x = \xi + 2t + \frac{1}{2}t^2$

$2t + \frac{1}{2}t^2 < x$

Ch. from $x = \xi$, $t = 0$, $\xi < 0$

$u = 1 + t$

$x = \xi + t + \frac{1}{2}t^2$

$t + \frac{1}{2}t^2 > x$

In gap

$u = a + t$ where $1 < a < 2$

$x = at + \frac{1}{2}t^2$

Region

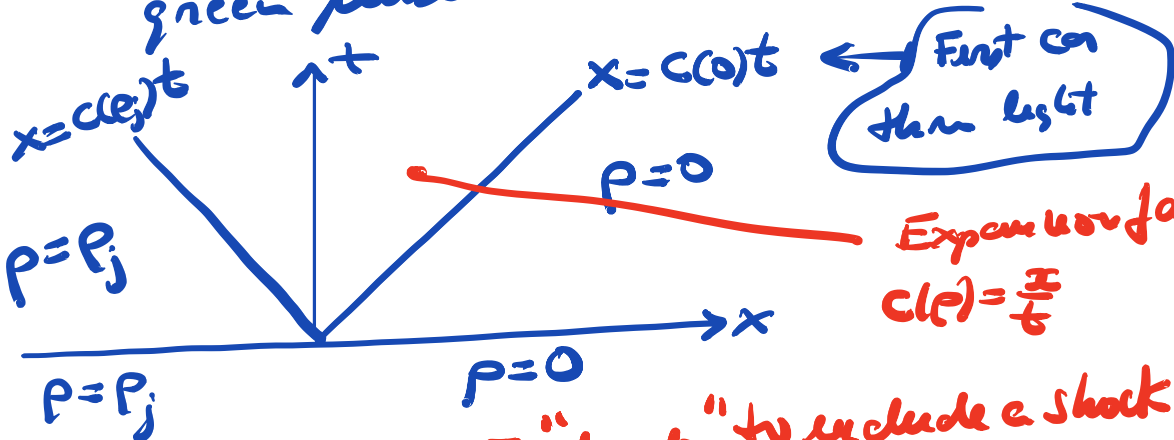
$t + \frac{1}{2}t^2 < x < 2t + \frac{1}{2}t^2$

$a = \frac{1}{t}\left[x - \frac{1}{2}t^2\right]$

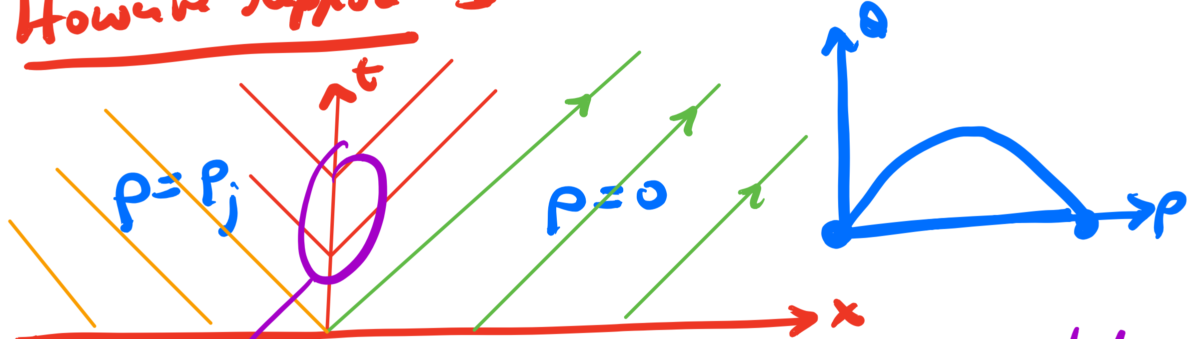
$u = \frac{1}{t}\left(x - \frac{1}{2}t^2\right) + t$

Example 5 Why Lax entropy matters

Red light turns green problem done "ignoring"



However suppose I "decide" to include a shock



note: information is being generated at this "fake" shock!!

where is it coming from?

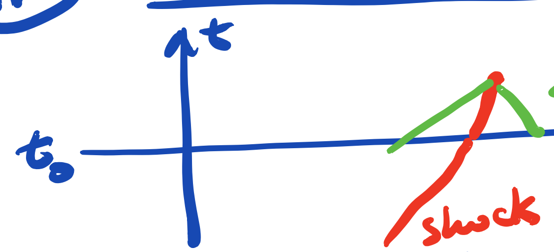
Conclusion: lose uniqueness!! problem becomes ill-posed

Many more such sols. allowed

Argument to justify Lax Entropy!!

#1 Use shock only when needed, to stop characteristics from crossing.
"do simplest fix to multiple values"

#2 Want a well posed problem



ch. allow you to compute value of ρ on both sides of shock!

Know sln for $t<t_0$
want to compute it at t_0+dt

at next time need to know sln. and where shock moves to

then use R.H. to get shock speed and you are done

If ch. don't converge you have a problem

#3 physical argument

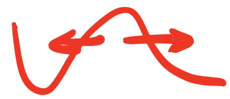
Shock arises from competition between

nonlinearity and some "diffusion" effect [Higher order]

Example Traffic flow $\rho_t + q(\rho)_x = \nu \rho_{xx}$

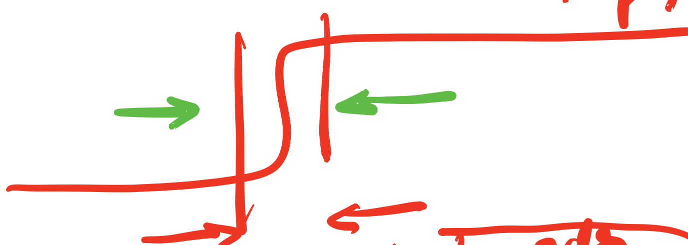
(preventive driving)

"Something that "wants" to speed things"



"shock" physical

Example



layer where higher order effect operates

This region will remain thin only if nonlinearity keeps narrowing it

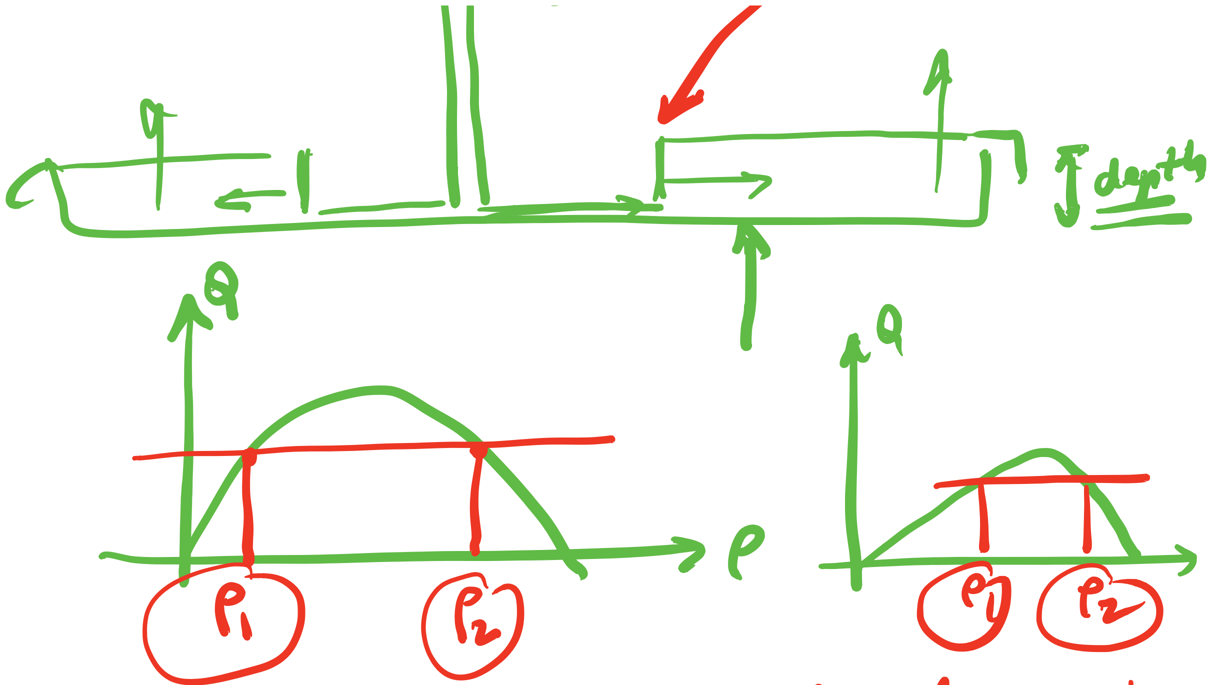
---

End

Example of controlling "where" a shock sits with parameters

|| Flow rate

hydraulic jump



Generate steady shockwave traffic flow with incoming density ρ_1 by using a "leaky" red light that produces ρ_2 behind (not ρ_j). An example of a "leaky" red light is <u>toll booths</u>.