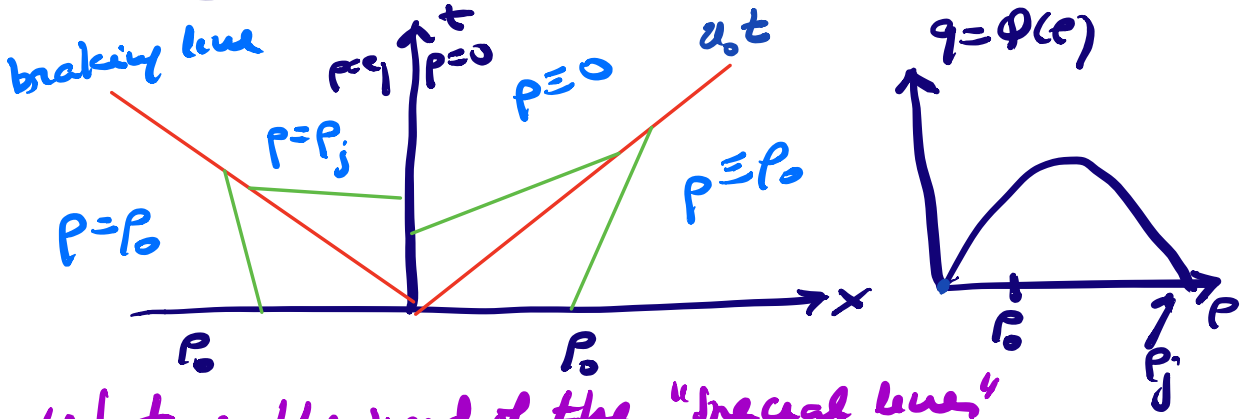
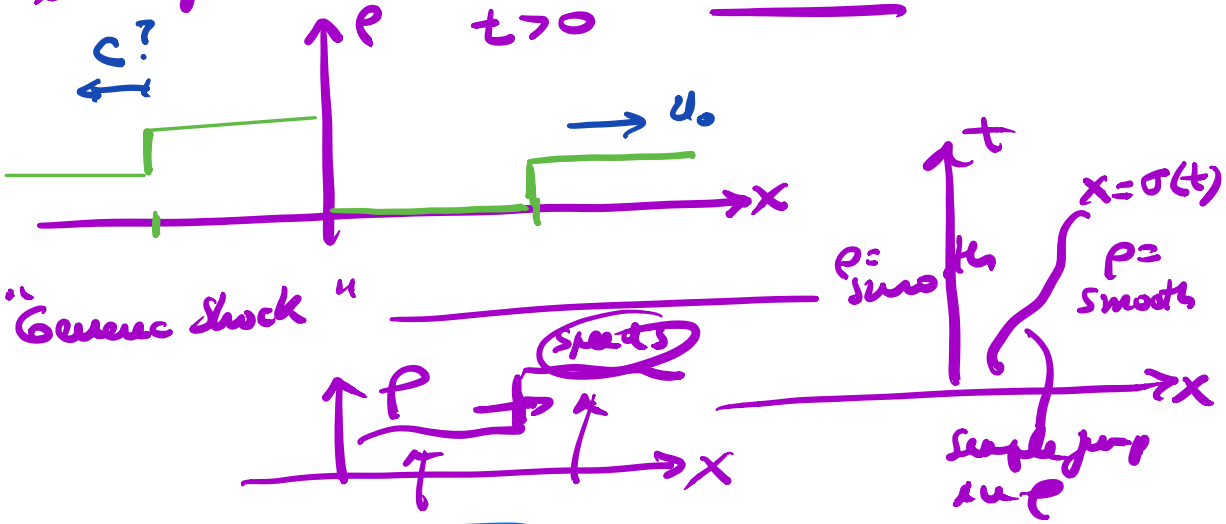
Lecture 12 Tue April 6, 2021

Green light traffic read problem $\rho_t + q_x = 0$



What is the speed of the "special lines" in space-time

"shocks"



$$\frac{d}{dt}\int_a^b \rho\, dx = q_a - q_b \qquad a < \sigma < b$$

$$q_a - q_b = \frac{d}{dt}\left(\int_a^\sigma \rho\, dx + \int_\sigma^b \rho\, dx\right) =$$

$\rho_t + q_x = 0$ where ever ρ has derivatives

$$= \int_a^{\sigma} \rho_t \, dx + \dot{\sigma}\rho\Big|_{\sigma}^{-} + \int_{\sigma}^{b} \rho_t \, dx - \dot{\sigma}\rho\Big|_{\sigma}^{+}$$

$= -q_x$ $\qquad$ $= -q_x$

$$= -q^- + q_a + \dot{\sigma}\rho^- - q_b + q^+ - \dot{\sigma}\rho^+$$

$+$ means evaluate at $\sigma+0$

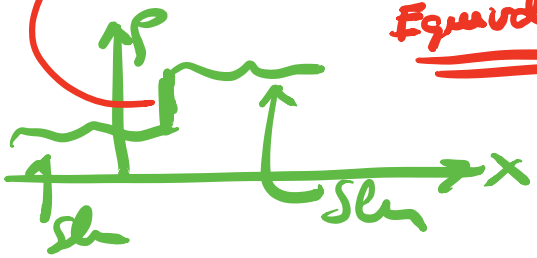
$-$ means evaluate at $\sigma-0$

$$\Rightarrow \quad 0 = -q^- + q^+ + \dot{\sigma}(\rho^- - \rho^+)$$

$$\therefore \quad \boxed{\dot{\sigma} = \frac{q^- - q^+}{\rho^- - \rho^+} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{[q]}{[\rho]}}$$

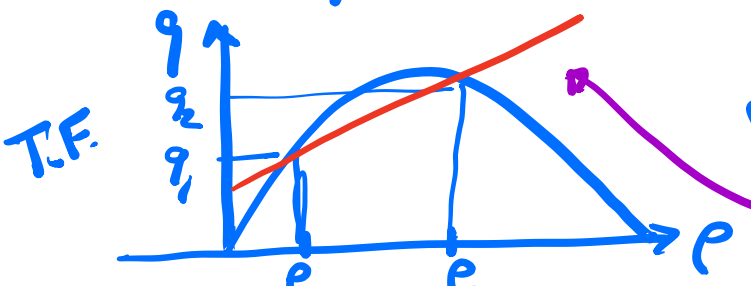
Rankine-Hugoniot Jump Conditions

Equivalent conservations



#1 Infinitesimal shock speed is characteristic speed: $\frac{dq}{d\rho}$

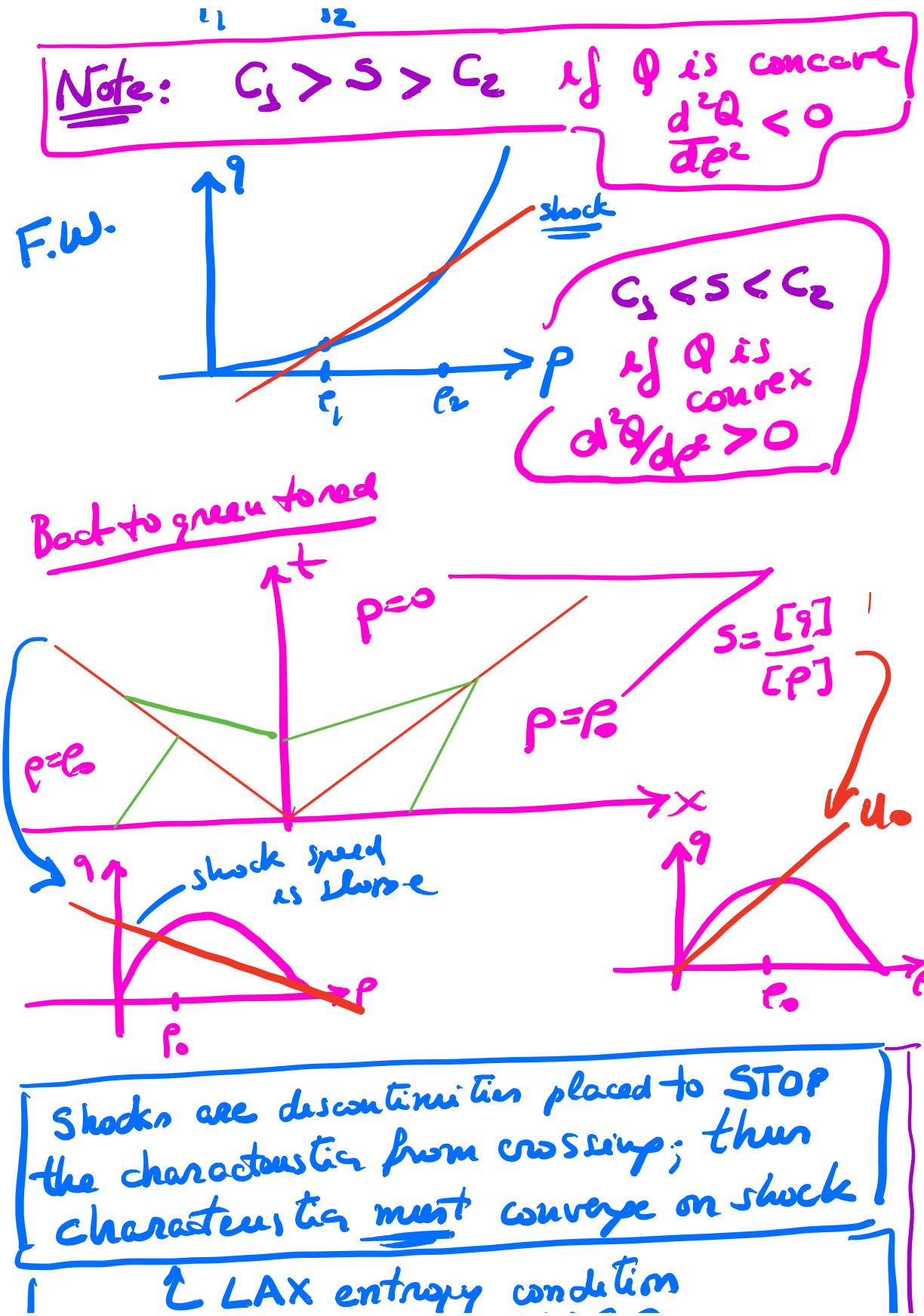
#2 Geometrical interpretation



ρ_1 = value on left

ρ_2 = value on right

shock speed = slope of secant


Note: $c_1 > s > c_2$ if Q is concave $\frac{d^2Q}{d\rho^2} < 0$
F.W.
q
shock
ρ
ρ_1
ρ_2
$c_1 < s < c_2$
if Q is convex
$d^2Q/d\rho^2 > 0$
Back to green to red
t
$\rho=0$
$\rho=\rho_0$
$\rho=\rho_0$
$s=\frac{[q]}{[\rho]}$
x
q
shock speed as slope
ρ
ρ_0
u_0
q
ρ
ρ_0
Shocks are discontinuities placed to STOP the characteristics from crossing; thus characteristics must converge on shock
↑ LAX entropy condition

Math characteristic speed on the left
> shock speed
> ch. speed on the right

For q concave ρ increases across shock

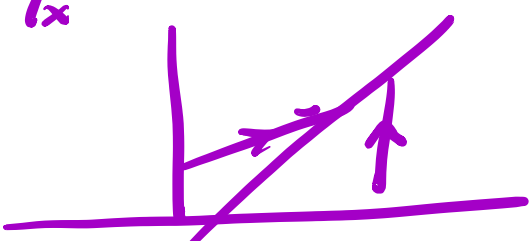
For q convex ρ decreases across shocks

→ Traffic Flow shocks

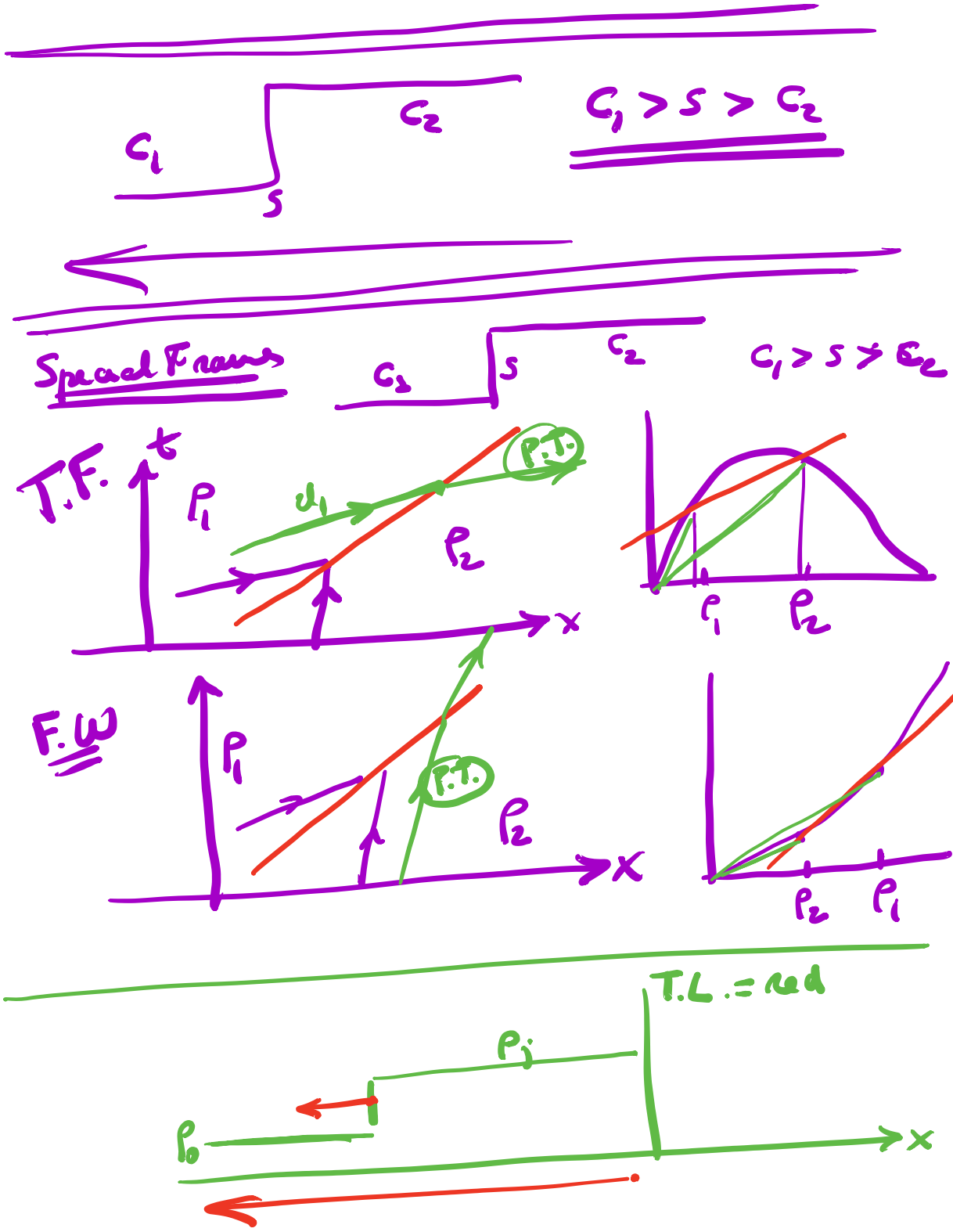
Flood wave shocks

Hydraulic jumps

$\rho_t + q_x = 0$ with no shock is "reversible"



but shocks make it [non-reversible]

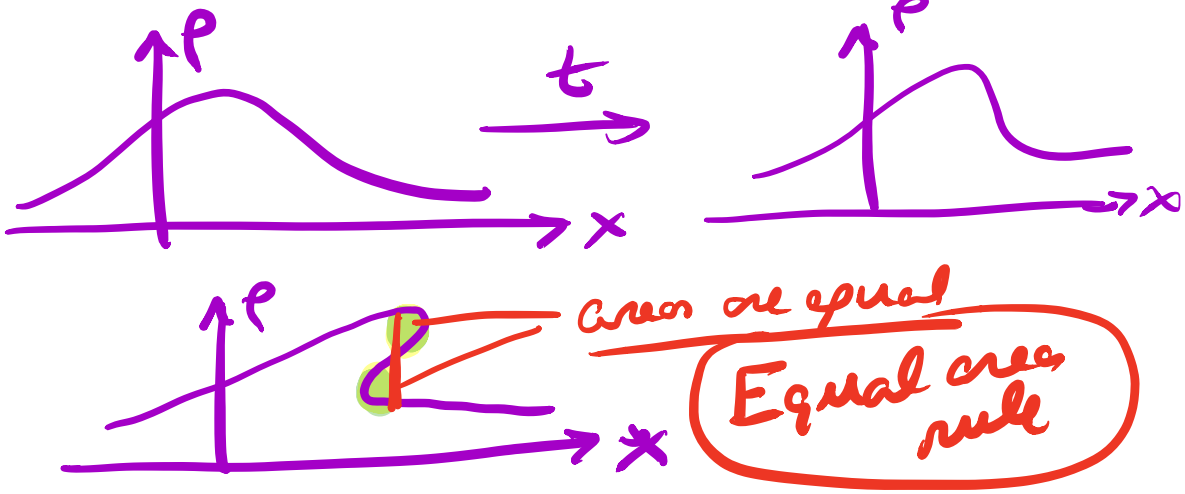

c_1
c_2
S
$c_1 > s > c_2$
Spread Frame
c_2
S
c_2
$c_1 > s > c_2$
T.F.
t
p_1
u_1
P.T.
p_2
x
p_1
p_2
F.W
p_1
P.T.
p_2
x
p_2
p_1
T.L. = nd
p_j
p_0
x

Extended Model

① Soln satisfies pde where smooth
$R_t + q_x = 0$

② Shocks are introduced to stop crossing of ch. / multiple values

Shocks Satisfy Rankine-Hugoniot + Entropy



Note This depends on a conservation law

Need conservation for shocks

How to be "physical" conservation

$u_t + uu_x = 0$ → $u_t + (\frac{1}{2}u^2)_x = 0$ ∴ Cons. of u

$(\frac{1}{2}u^2)_t + (\frac{1}{3}u^3)_x = 0$ (cons. of u^2)

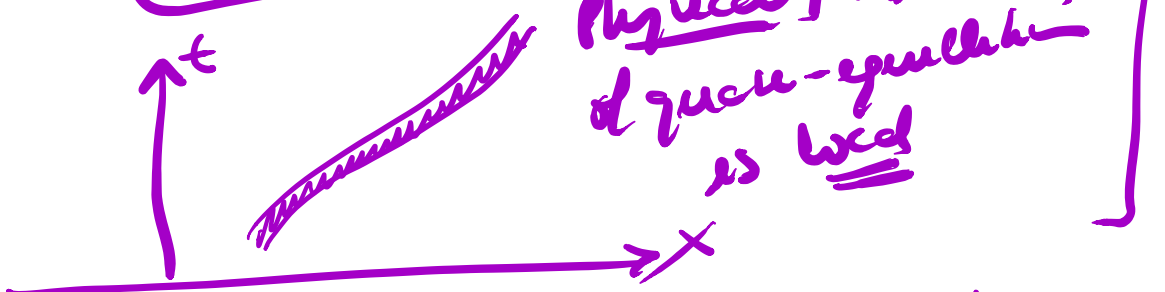
Example In Gas Dynamics (Euler Eqns)

smooth sln conserve mass ✓ momentum ✓ energy ✓

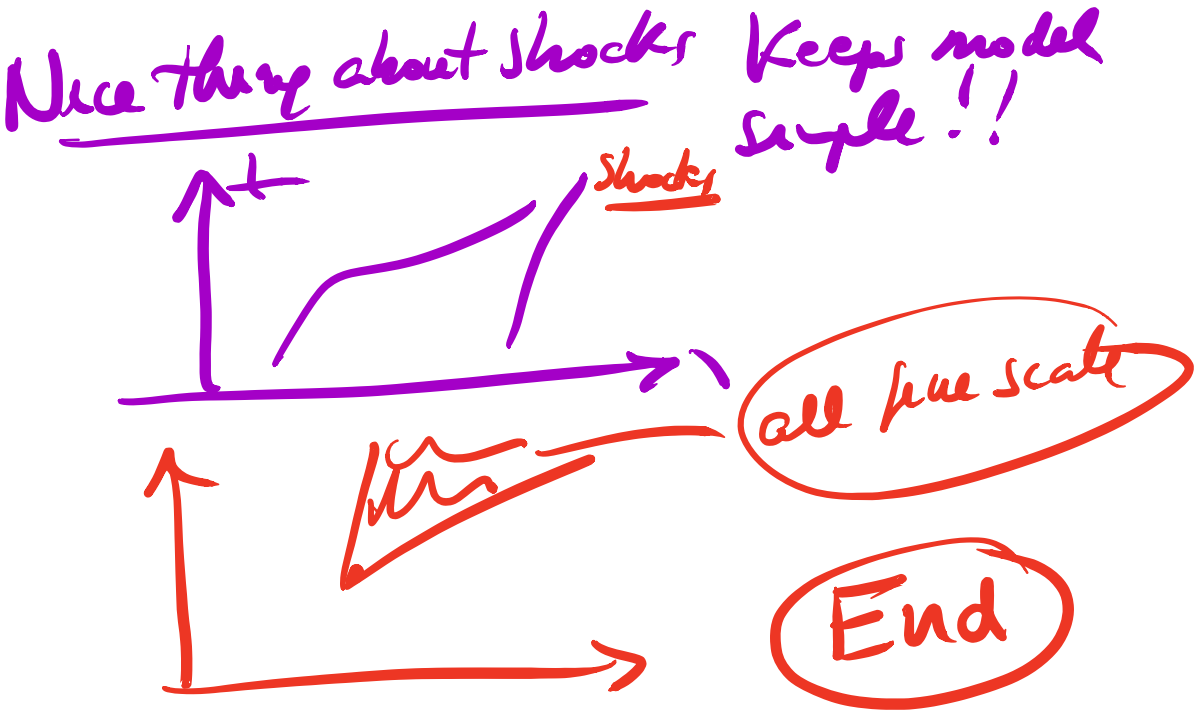
and entropy! S = entropy $\quad (\rho S)_t + (\rho u S)_x = 0$

Note 2 Shocks are ONLY one possible sln of the multiple valued problem!

Must look at physics before placing shocks



Plasma "Collisionless shock"


Nice thing about shocks
Keeps model simple!!
Shocks
all time scale
End