Lecture 10, March 30, 2021

Quasilinear pde's

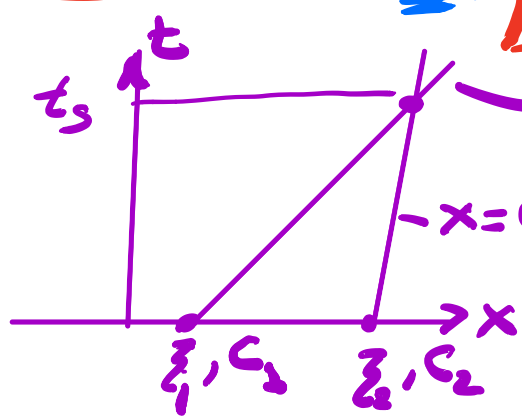
$u_t + c(u)u_x = 0$; $u(x,0) = F(x)$

Char. form $\frac{dx}{dt} = c(u)$ and $\boxed{\frac{du}{dt} = 0}$

$x = \xi$ at $t=0$ $\quad u = F(\xi)$ at $t=0$

$$\boxed{\begin{aligned} x &= c(F(\xi))t + \xi = X(\xi, t) \\ u &= F(\xi) \end{aligned}}$$

def $\boxed{C(\xi) = c(F(\xi))}$

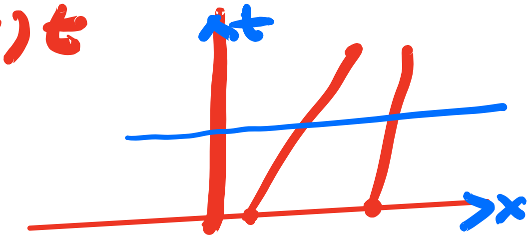


Case $c_1 > c_2$

Need $\frac{dC}{d\xi} < 0$ somewhere

$X(\xi, t) = \xi + C(\xi)t$

$$\boxed{X_\xi = 1 + tC'(\xi)}$$



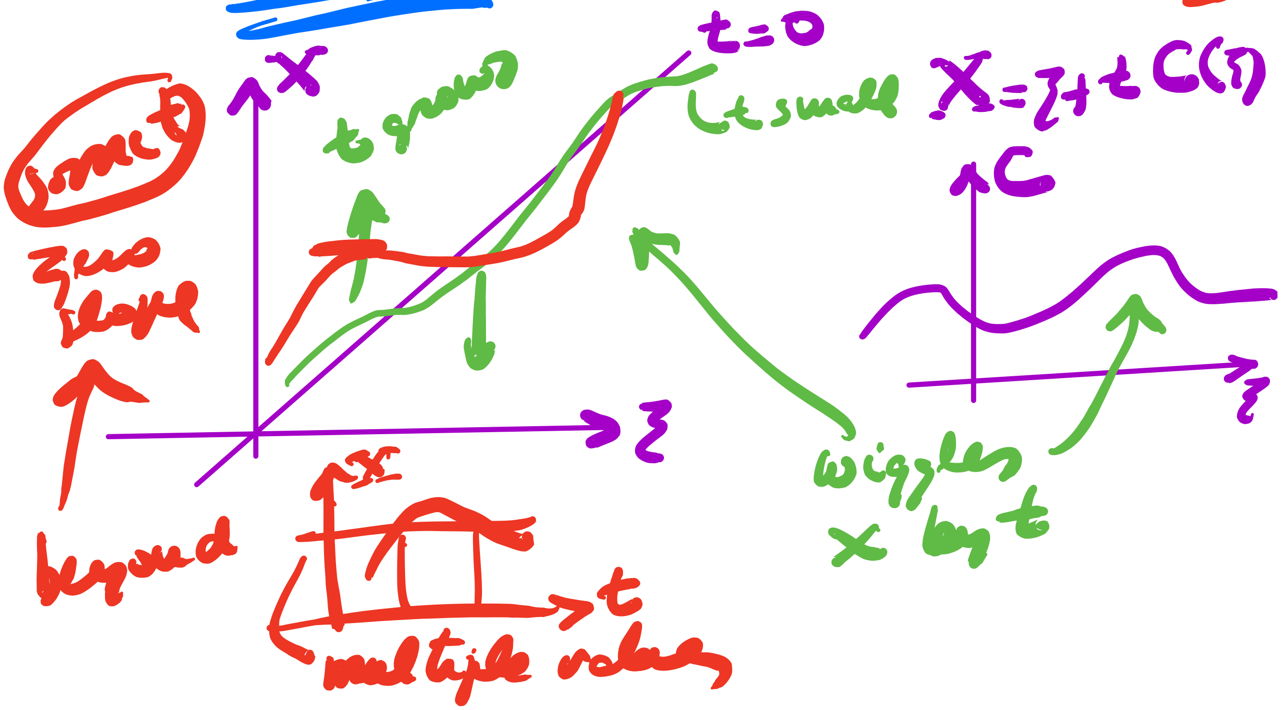
As long as $C'(\xi)$ is bounded

for t small enough $X_\xi > 0$

i.e. X is increasing so

it has inverse $\xi = \xi(x,t)$

unique



$X' = 1 + t\,C'(\xi)$ When do you

hit critical (i.e. cannot

invert $\xi = \xi(x,t)$) Need

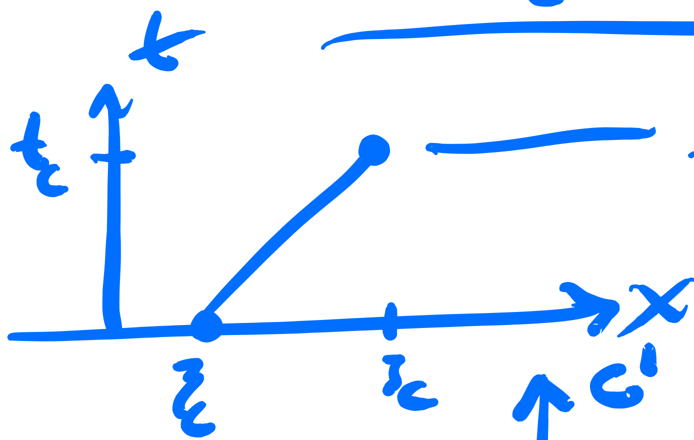
$X_\xi = 0$

a) C' must < 0 somewhere

b) Take $\min_{\text{over } \xi} C' = -\frac{1}{t_c}$, happens at $\xi = \xi_c$

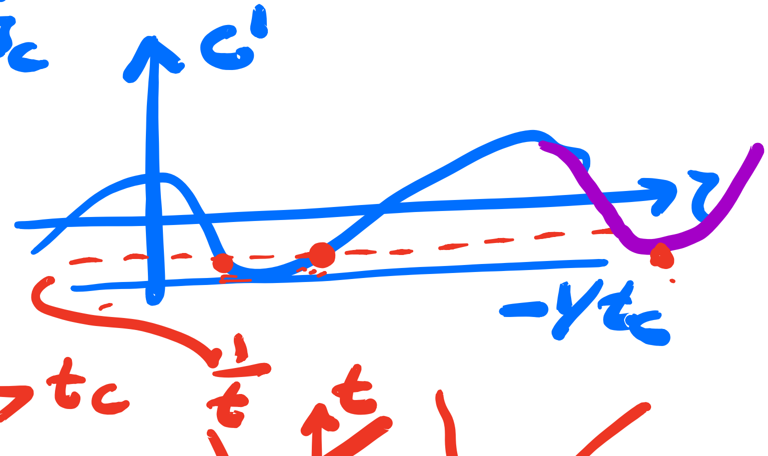
$$\boxed{X_\xi = 1 + t\,C'(\xi) = 0} \quad \text{at } \xi = \xi_c,\ t = t_c$$

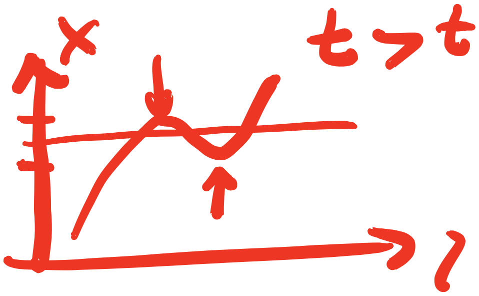
$$x_c = \xi_c + t_c\, C(\xi_c)$$



$\frac{dX}{d\xi} = 0$ for first time

Graphically





Geometry of Ch.

Envelope of family of curves — Curve made up by intersections of infinitesimal neighbours

property Envelope = curve such that every point belongs to a member of the family and is tangent to the member there

Example (abstract)

Family of curves $F(x,t,\xi)=0$

Def. #1 $F(x,t,\xi)=0$

$F(x,t,\xi+d\xi)=0$

$\cancel{F(x,t,\xi)} + F_\xi(x,t,\xi)\,d\xi$

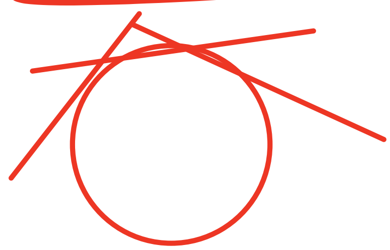
equiv $\boxed{F = F_\xi = 0}$

Example $(\cos\theta)x + (\sin\theta)y - 1 = 0$

$\Longrightarrow$ Envelope + $-(\sin\theta)x + (\cos\theta)y = 0$

$\Rightarrow (x,y) \propto (\cos\theta, \sin\theta)$

i.e. $x = \cos\theta,\ y = \sin\theta$



Def. #2 If (x,t) is an envelope

then $F(x,t,\xi) = 0$

for some ξ

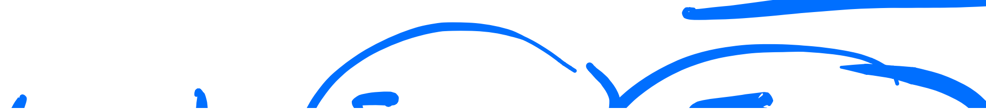
and at ξ curve and member

have the same tangent

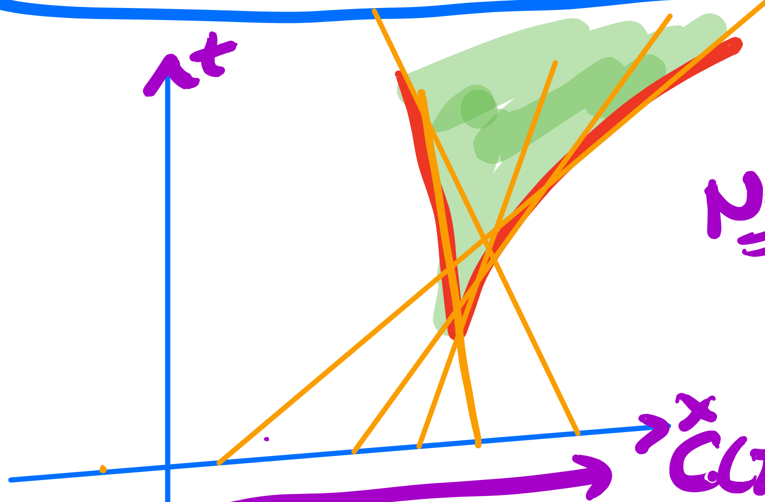
$X = X(\xi)$ $\quad F(X,Y,\xi) = 0$

$Y = Y(\xi)$ $\quad X', Y' \propto$ Tangent F_0 Family member

$dx\,F_x + dy\,F_y = 0 \quad \rightarrow \quad X'F_x + Y'F_y = 0$



$$x'F_x + y'F_y + F_\tau = 0 \qquad \boxed{F_\tau = 0}$$



Note $c' > 0$ not an assumption

$C(\tau)$ decreasing

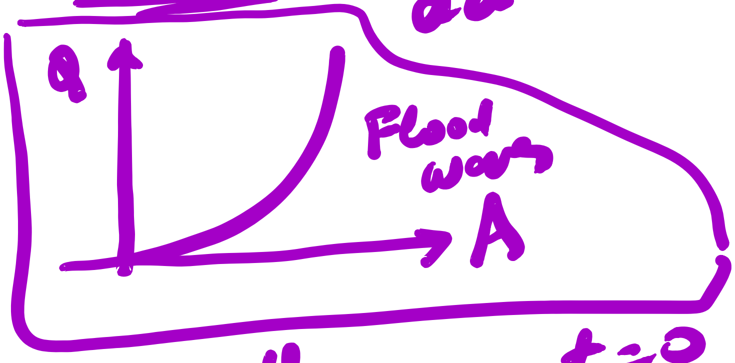
$$x = C(\tau)t + \tau \qquad C(\tau) = c(F(\tau))$$

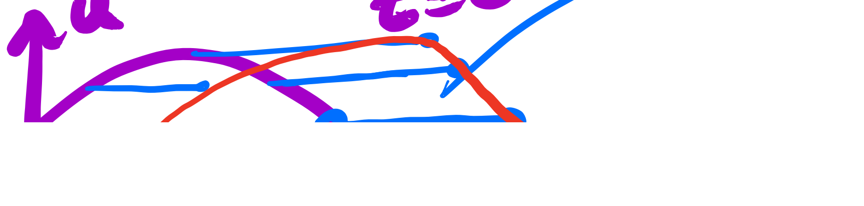
$$u = F(\tau)$$

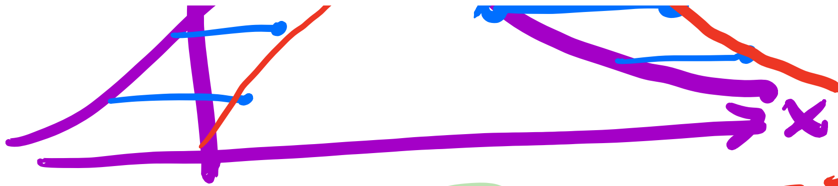
What is u doing??

Assume $\dfrac{dc}{du} > 0 \quad \therefore \quad C' > 0 \Leftrightarrow F' > 0$

$C' < 0 \Leftrightarrow F' < 0$







steeper

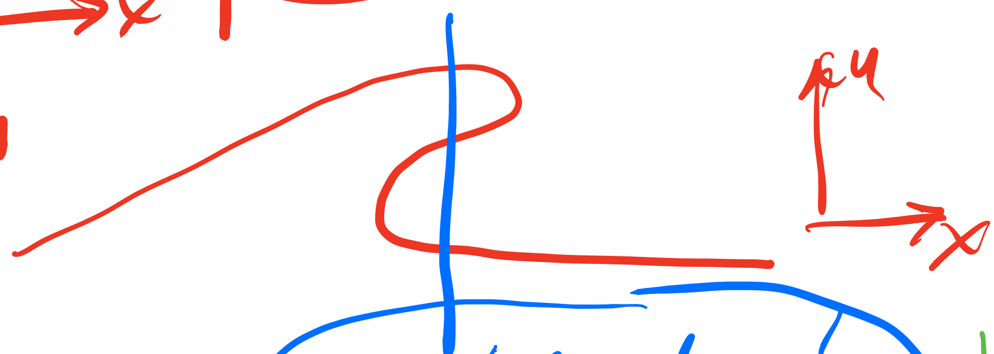
Wave steepening

shallower

infinite slope d at some time

beyond

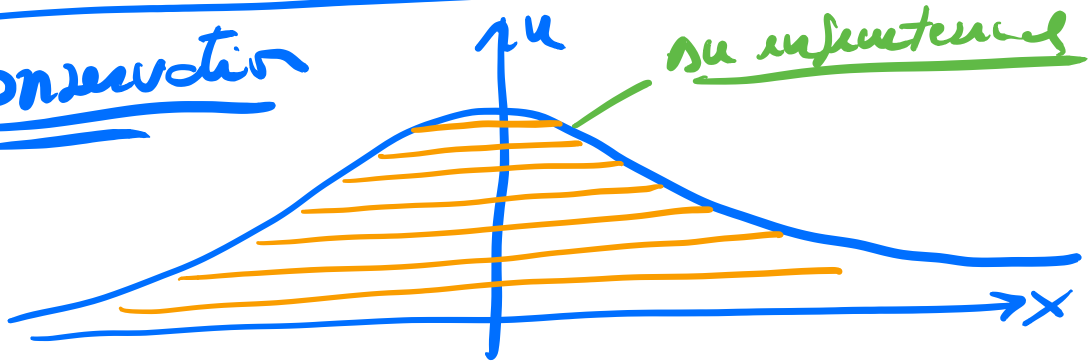
t_c



multiple values!

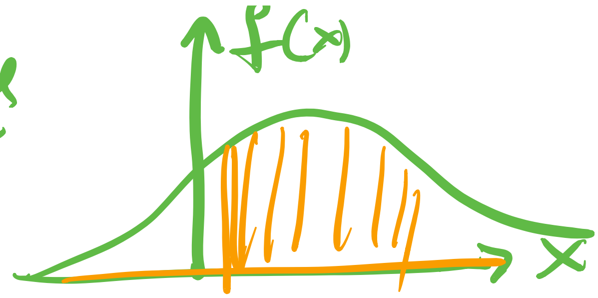
Conservation

an infinitesimal



Stop

Riemann Integral



Goes back to Aristotle

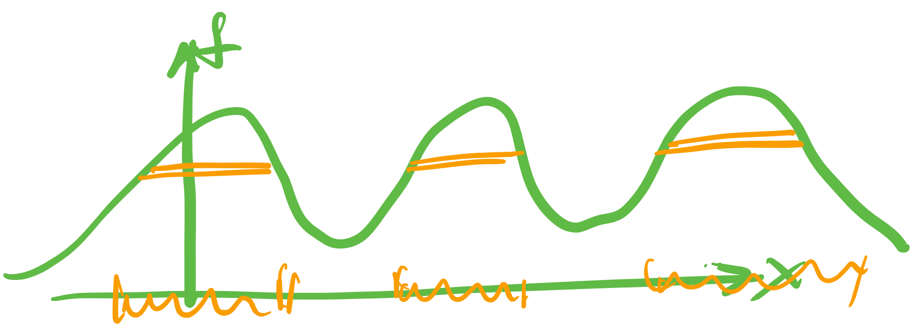
quadratic

$$\lim \int f_n(x)\,dx \overset{?}{=} \int \lim f_n(x)\,dx$$

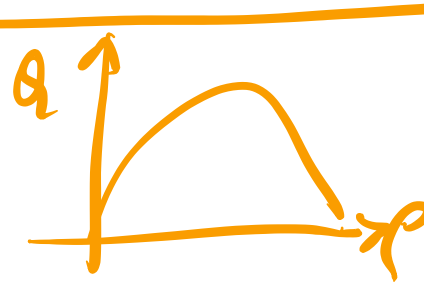
Very cumbersome with R. Int.

Lebesgue

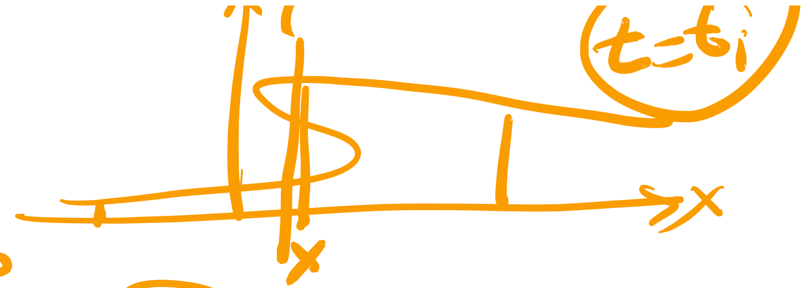
Lebesgue Int. ~1900



$\rho_t + q_x = 0 \quad q = Q(\rho)$



$\frac{dQ}{d\rho}$ decreasing



at $x = x_1$