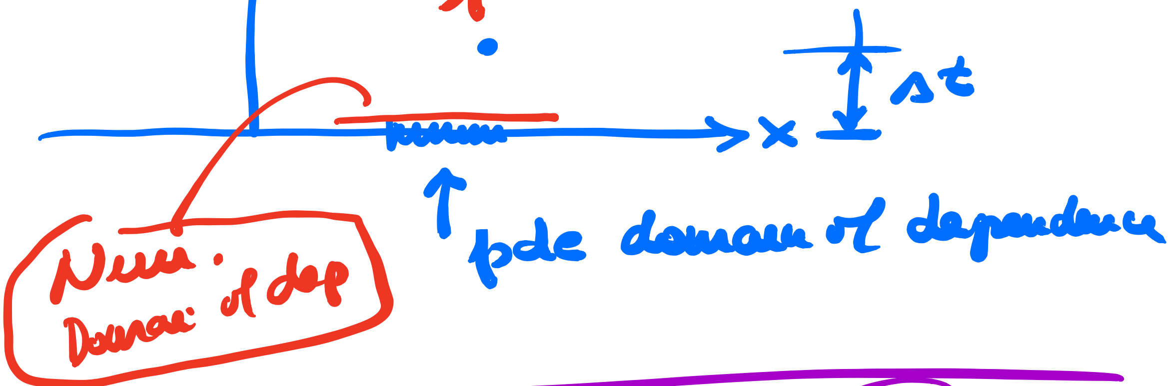
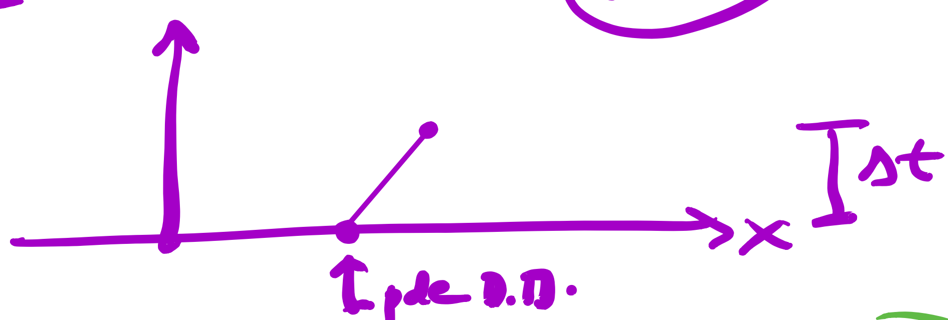
Lecture 9, Th March 25, 2021

CFL condition & related

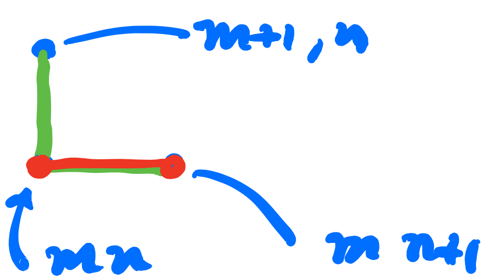
[Num. Domain dependence ≥ pde Domain dependence]

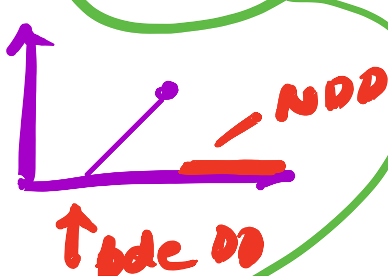


Example $u_t + u_x = 0$ $\qquad \frac{dx}{dt} = 1$



#1 Forward diff $\quad \dfrac{u_n^{m+1} - u_n^{m}}{\Delta t} + \dfrac{u_{n+1}^{m} - u_n^{m}}{\Delta x} = 0$

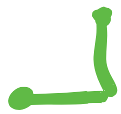


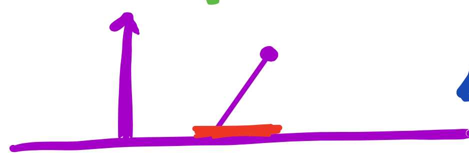


Will not work

Backward diff

replace by $\frac{u_n^m - u_{n-1}^m}{\Delta x}$

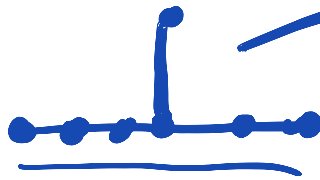




$\Delta t < \Delta x$
O.K.

General

Stencil



CFL

$u_t + c u_x = 0$

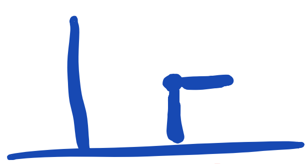
$$\boxed{\Delta t < \frac{\alpha}{|c|} \Delta x}$$

Necessary but not sufficient!!

Implicit

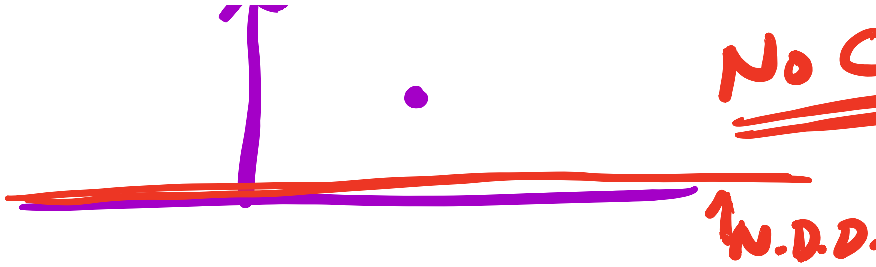
Backward Euler

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} + \frac{u_{n+1}^{m+1} - u_n^{m+1}}{\Delta x} = 0$$



$$u_n^{m+1} + \frac{\Delta t}{\Delta x} u_{n+1}^{m+1} - \frac{\Delta t}{\Delta x} u_n^{m+1} = u_n^m$$

$$A\vec{u}^{m+1} = B u^m$$



No CFL

$\vec{u}^{m+1} = F(\vec{u}^{m})$ Explicit

$G(\vec{u}^{m+1}) = F(\vec{u}^{m})$ Implicit

$\vec{u}^{m+1} = F\vec{u}^{m}$

$G\vec{u}^{m+1} = F\vec{u}^{m} \longleftrightarrow \boxed{\vec{u}^{m+1} = G^{-1}F u^{m}}$

$u_t = \mathcal{L}u \qquad u = e^{\lambda t}\varphi(\vec{x})$

Linear $\qquad \boxed{\lambda\varphi = \mathcal{L}\varphi}$

$\mathcal{L}u = u_x \qquad \mathcal{L}u = u_x + u_{xxx}, \qquad \mathcal{L}u = u_{xx},$

$\mathcal{L}u = \Delta u, \ \ldots$

$\mathcal{L}(e^{\lambda t}\varphi) = e^{\lambda t}\mathcal{L}\varphi$

$u_t = u_{xx}$

$0 < x < \pi$

$u\big|_{x=0} = 0 \qquad u\big|_{x=\pi} = 0$

$\lambda\varphi = \varphi_{xx} \quad \varphi(0) = \varphi(\pi) = 0$

$\varphi = \sin(\alpha x) \quad \alpha = \text{integer}$

$\boxed{\lambda = -n^2 \quad \varphi = \sin nx}$

$$\boxed{u = \sum u_n e^{-n^2 t} \sin nx}$$

Initial data $= \sum a_n \sin nx$

$$\boxed{\vec{u}^m = \sum \alpha_n G_n^m \vec{\varphi}_n}$$

$$\vec{u}^0 = \sum \alpha_n \vec{\varphi}_n$$

$\vec{u}^{m+1} = A\vec{u}^m \quad \vec{u}^m = G^m \varphi$

$\boxed{G\varphi = A\varphi} \quad |G| \leq 1$

Eigenvalue problem

Matrix (Square)

$\boxed{Av = \lambda v}$

$A = n \times n$

$$0 = \det(A - \lambda) = p(\lambda)$$

$\underbrace{}_{\text{degree = size}}$ A $= n$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1$ multiplicity 2

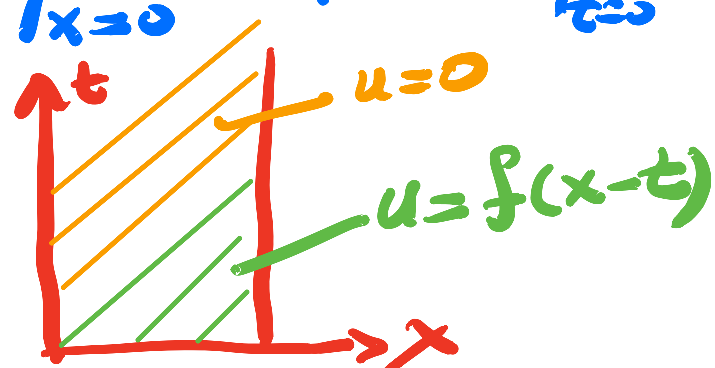
eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ just one

$A = A^{+} \implies$ eigenvalues are real

eigenvectors form complete orthogonal basis

$u_t + u_x = 0$ $u\big|_{x=0} = 0$ + I.C. $u\big|_{t=0} = f(x)$

Charact



Sep. Variable $u = e^{\lambda t}\varphi(x)$

$\lambda\varphi + \varphi' = 0$ & $\varphi(0) = 0$

$\varphi = a e^{-\lambda x}$ $\quad \varphi = a = 0$

NO eigenvalues! Sep. Variable FAILS

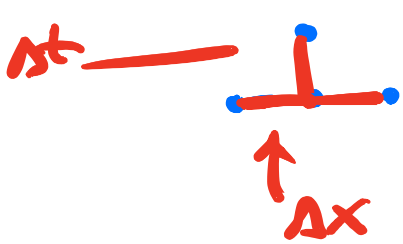
NO eigenvectors! Catastrophically

Back to "CFL"

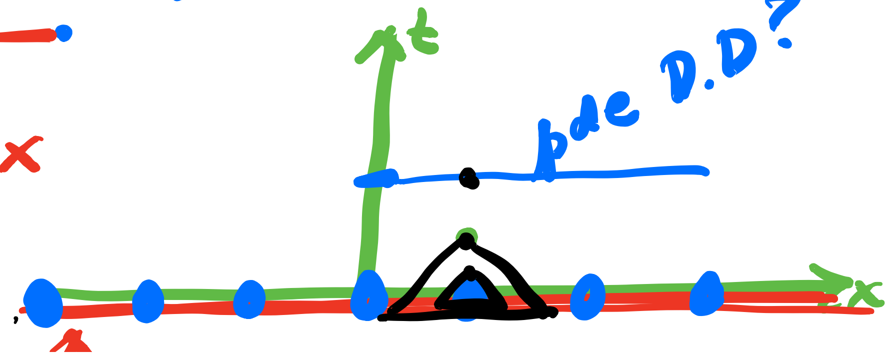
$\boxed{u_t = u_{xx}}$ periodic period 2π

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}$$

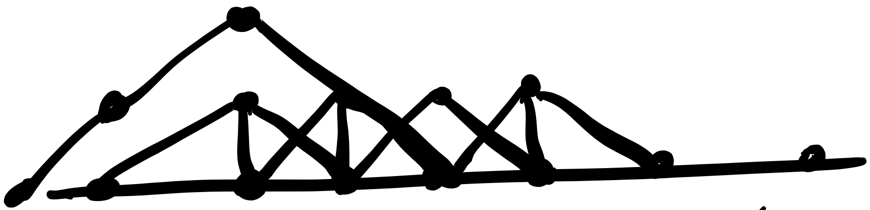
von Neumann $\Rightarrow$ Stability requires $\Delta t \le (\Delta x)^2$



$\leftarrow$ stencil



1pde D.D.



$$\boxed{u_t = u_{xxx}}$$

KdV

Explicit scheme

$$\Delta t \le (\Delta x)^3$$

Example Numerics

Quasilinear $\boxed{u_t + c(u)u_x = 0}$

e.g. Traffic Flow / Flood Waves etc

"Baby Gordywave's"

$$\frac{dx}{dt} = c(u) \qquad \frac{du}{dt} = 0 \qquad u(x,0) = f(x)$$

$-\infty < x < \infty$

slow here

Charact. that starts at $x = \xi$ at $t = 0$

$u = f(\xi)$

$$\boxed{x_0 = c(f(\xi))t + \xi}$$

Infinite Dimensions

Standard problem $\quad Av = \lambda v$

A is $n \times n$ matrix
v is n vector

dimension is $\underline{n}$

Where do the eigenvectors "live"

They are in R^n !

PDE problem $\quad u_t = u_{xx}$, 2π periodic in x

$\Rightarrow \boxed{\lambda \varphi = \varphi_{xx}}$ separate variables

infinite dim. eigenvalue problem

$u = e^{\lambda t}\varphi(x)$

Why infinite dim

Where do e.v. "live" They

are periodic functions

How do I see it is infinite d.

$$\boxed{\varphi = \sum \varphi_n e^{inx} \quad \text{for some } \varphi_n}$$

$$\boxed{\varphi = \sum a_n \cos nx + b_n \sin nx}$$

$$\dot{Y} = AY \qquad Y = \sum a_n e^{\lambda_n t} Y_n$$