Lecture 7   1st order scalar eqns (1-D)

$u_t + c(x)u = f(x)$, with $u(x,0) = U_0(x)$

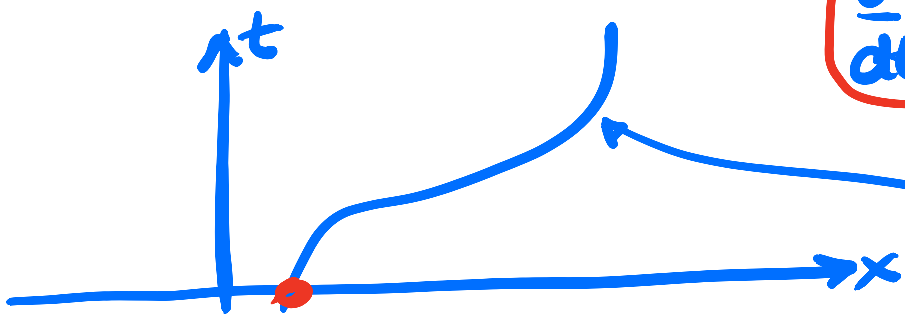
Sln. Charact. find $u = U(s,t)$, $x = X(s,t)$

$X_t = c(x)$, $X|_{t=0} = s$

$U_t = f(x)$, $U|_{t=0} = U_0(s)$

$\frac{dx}{dt} = c(x)$

$\frac{du}{dt} = f(x)$



$u_t + c(x)u_x = f(x) \longrightarrow \frac{dx}{dt} = c(x)$

$\rho_t + (u\rho)_x = 0 \Longrightarrow \rho_t + u\rho_x = -u_x\rho$

$u = u(x)$

Example $u_t + x u_x = 0$ $-1 < x < 1$

$u(x,0) = f(x)$

Question: do I need B.C.?

Answer NO!

Sln

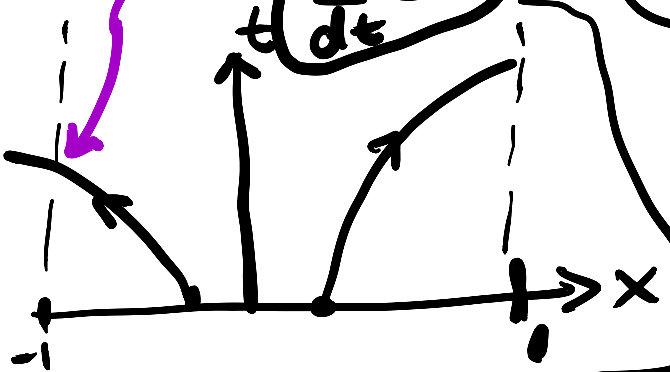
$$\frac{dx}{dt} = x \qquad \frac{du}{dt} = 0$$

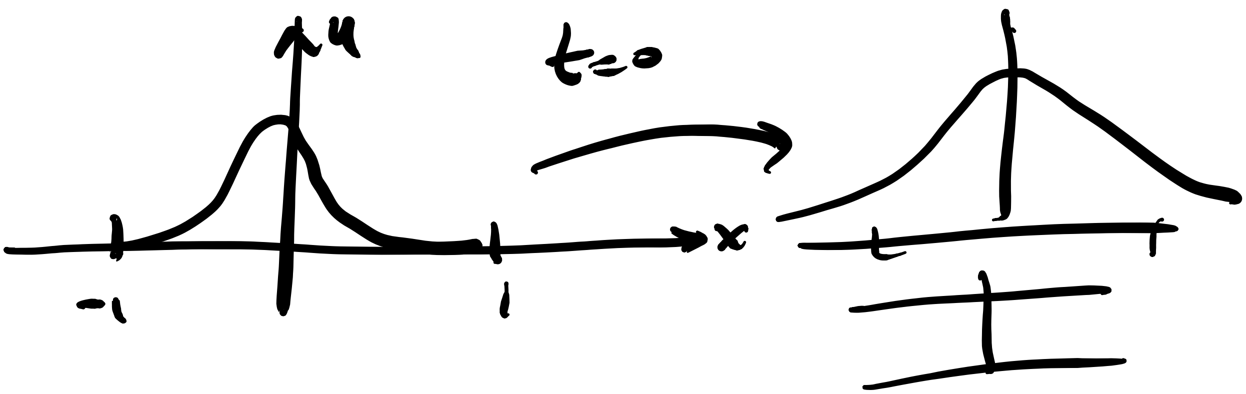
$$x = se^{t}$$

$$u = f(s)$$

$$s = xe^{-t}$$

$$\boxed{u = f(xe^{-t})}$$





Example #2

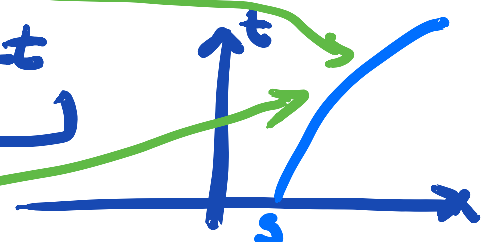
$$u_t + xu_x = -u$$

$$u_t + (xu)_x = 0$$

$$u(x,0) = f(x)$$

$$\frac{dx}{dt} = x \qquad x = se^{t}$$

$$\frac{du}{dt} = -u \qquad u = f(s)e^{-t}$$



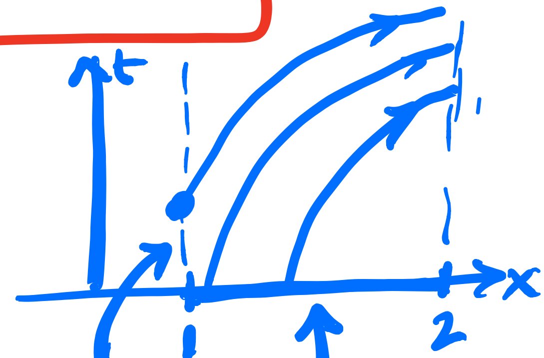
$$s = x e^{-t} \qquad u = f(x e^{-t}) e^{-t}$$

Example #3

$$\boxed{u_t + x u_x = 0 \quad 1 < x < 2 \\ u|_{t=0} = f(x)}$$

Now need B.C

Need B.C. at $x = 1$!!

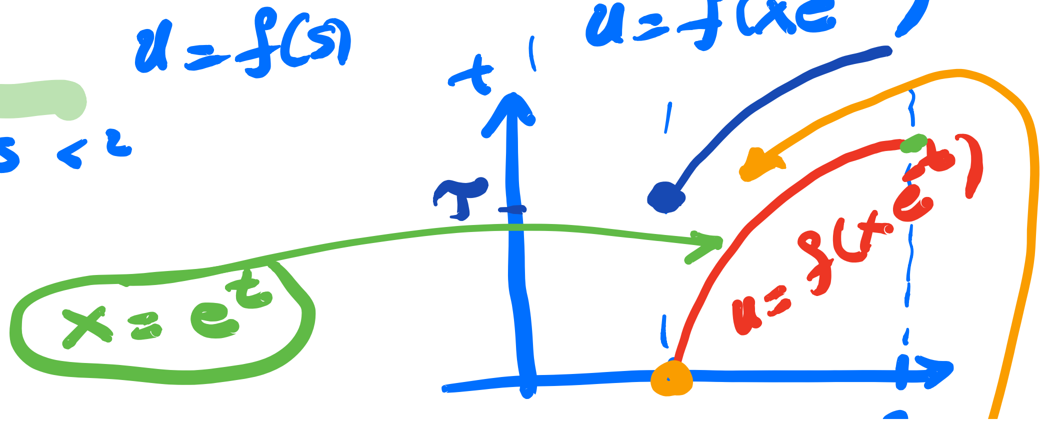


$u(1,t) = g(t)$ $\quad u = g(t) \quad u = f(x)$

$\frac{dx}{dt} = x$ and $x = s$ at $t = 0$ $\quad x = s e^{t}$

$u = f(s) \qquad u = f(x e^{-t})$

$1 < s < 2$



1 2

$$\frac{dx}{dt} = x$$

when $x = 1$, $t = T$

$$x = \cancel{c}\, e^{t}$$

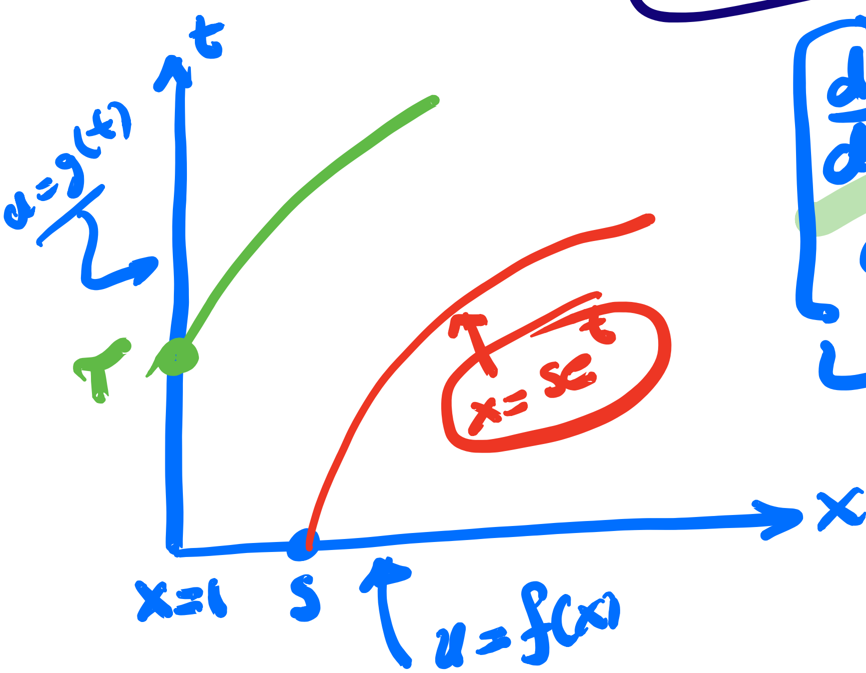
$$t - T = \ln x$$

$$x = e^{t-T}$$

$$u = g(T)$$

$$T = t - \ln x$$

$$\boxed{u = g(t - \ln x)}$$



$$\frac{dx}{dt} = x \quad \text{ch.}$$

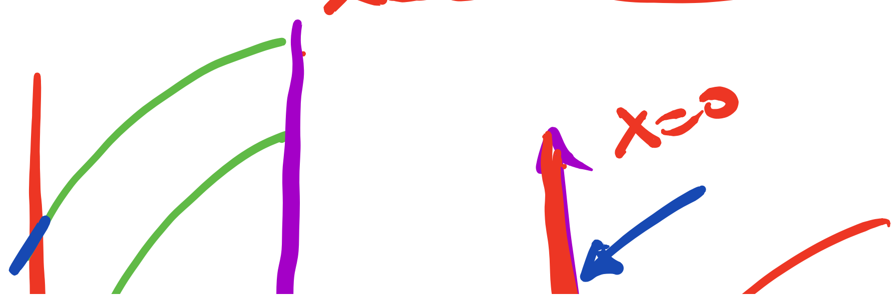
$$\frac{du}{dt} = 0$$

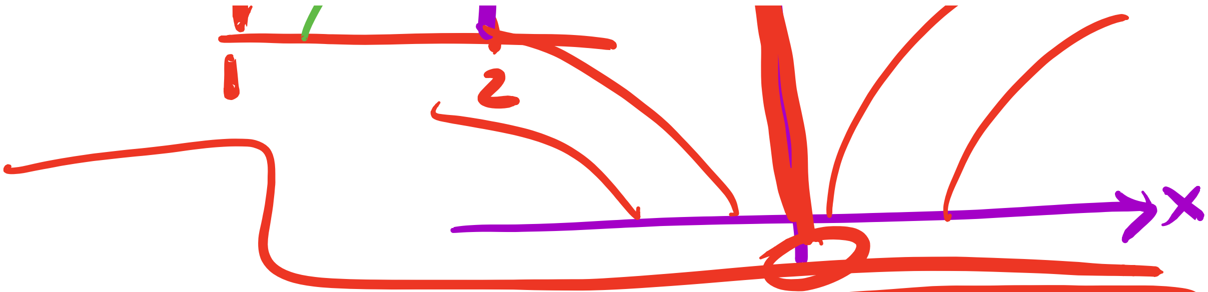
$$\frac{dx}{dt} = x$$

$x = s$ at $t = 0$

$\frac{dx}{dt} = x$ and when $t = T$, $x = 1$

$$x = e^{t-T}$$



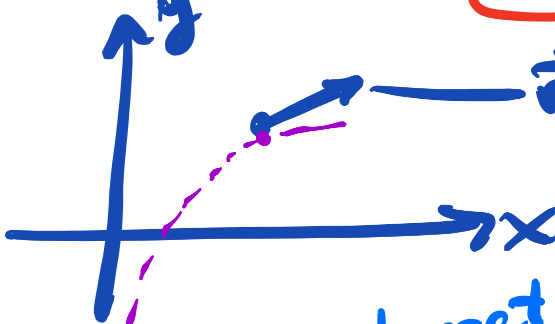


General set up

1-order linear scalar 2-D

$$a u_x + b u_y = cu + d$$

$a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$, $d = d(x,y)$



Charact.

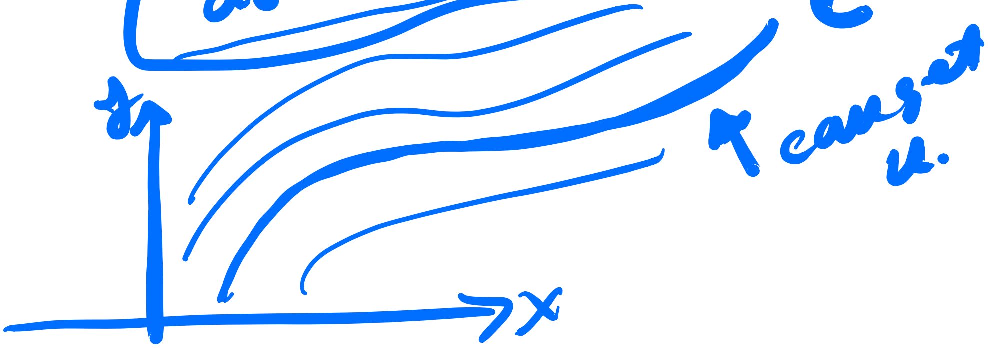
$$\frac{dx}{dt} = a$$

$$\frac{dy}{dt} = b$$

Eqn

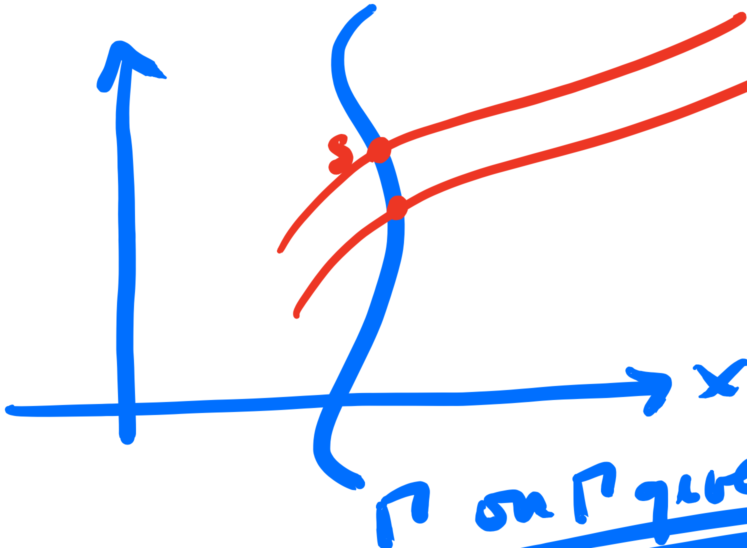
$$\frac{du}{dt} = cu + d$$

C



Info need to find u

Need to know u at **one** point along each charact.



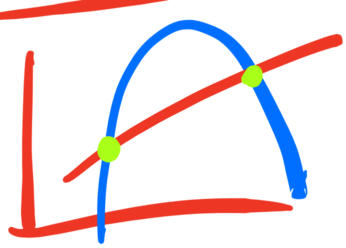
say s is parametr along Γ

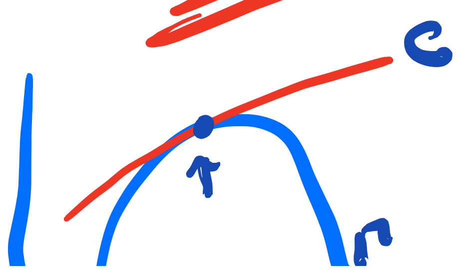
on Γ give u

$$\Rightarrow \boxed{\begin{aligned} x &= X(s,t) \\ y &= Y(s,t) \\ u &= U(s,t) \end{aligned}}$$

Cross charact. only **once** and cannot be **tangent** to them

Constraints on Γ



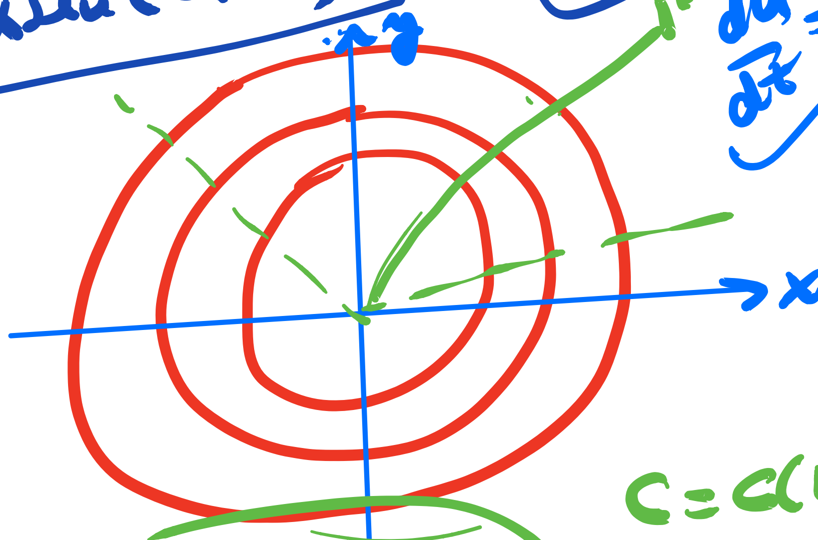


Take $a=-y$, $b=x$

$$\boxed{\begin{aligned} x &= \alpha\cos(t+t_0) \\ y &= \alpha\sin(t+t_0) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= a = -y \\ \frac{dy}{dt} &= b = x \end{aligned}}$$

$$\frac{du}{dt} = cu+d$$

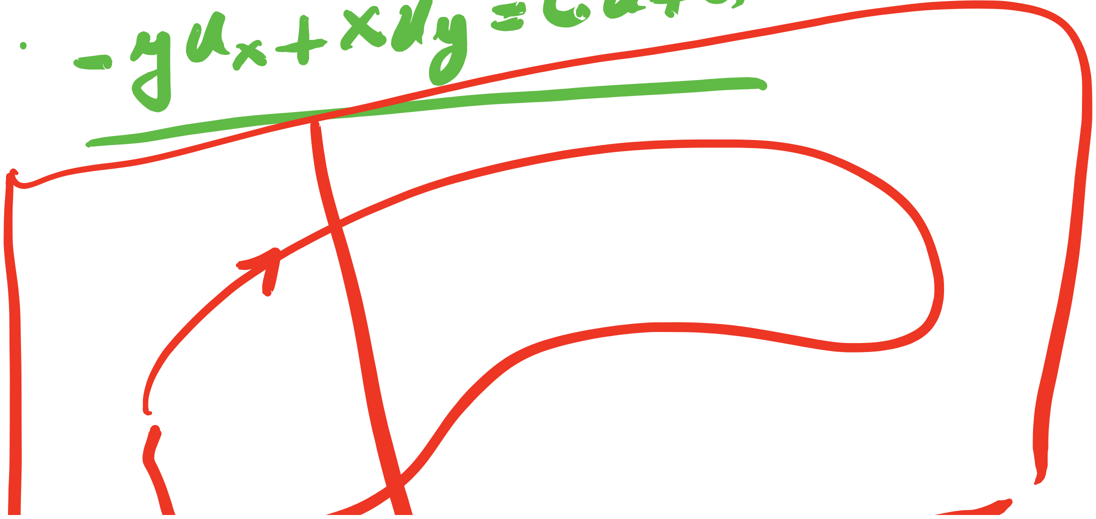


$c = c(r,\theta)$

$d = d(r,\theta)$

$$\frac{du}{d\theta} = cu+d$$

$$-yu_x + xu_y = cu+d$$



$-y u_x + x u_y = u$ $\quad r<1$

and $u(x,0)=f(x)$

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

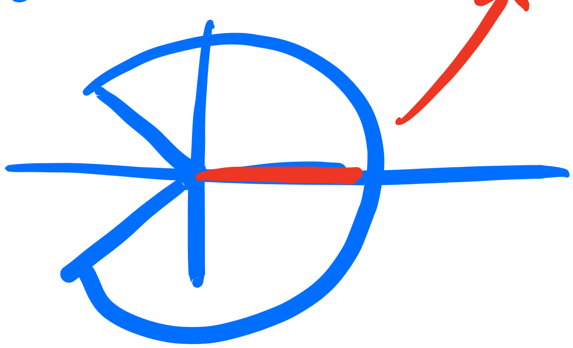
$$\boxed{\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}}$$

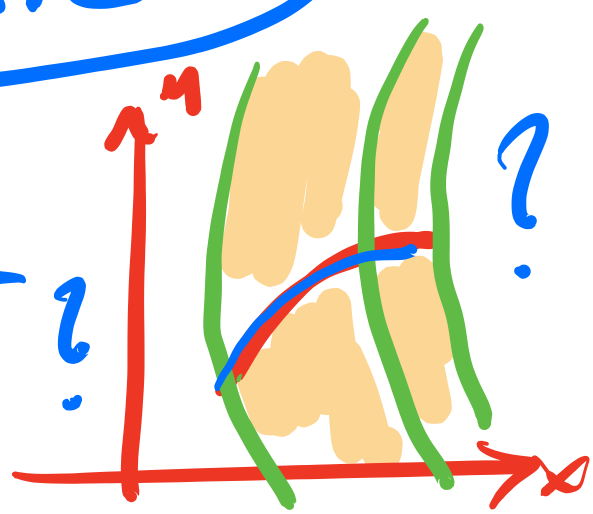
$t = \theta$

$\frac{du}{dt} = 0 \qquad u\Big|_{\substack{y=0 \\ x>0}} = g(x)$

$u = g(r)$ at $t=0$

$\frac{du}{dt} = u \qquad u = g(r)e^{\theta}$

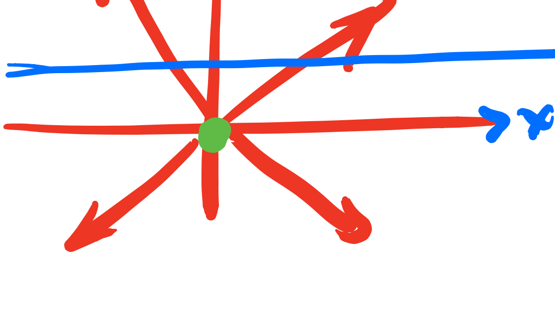




Domain of dependency

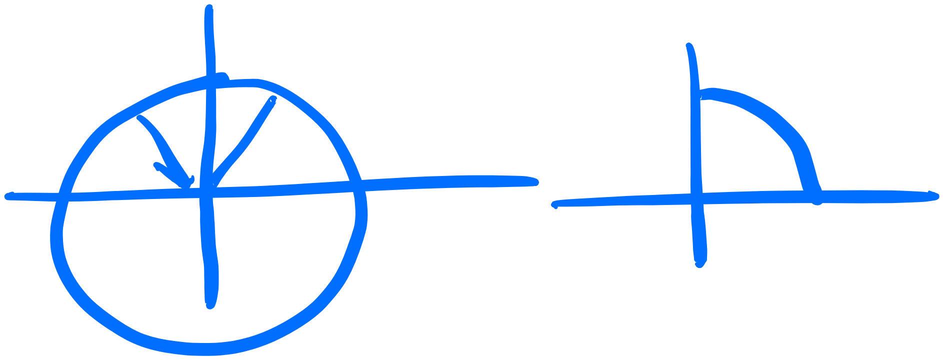
Example $\boxed{xu_x + yu_y = 0}$

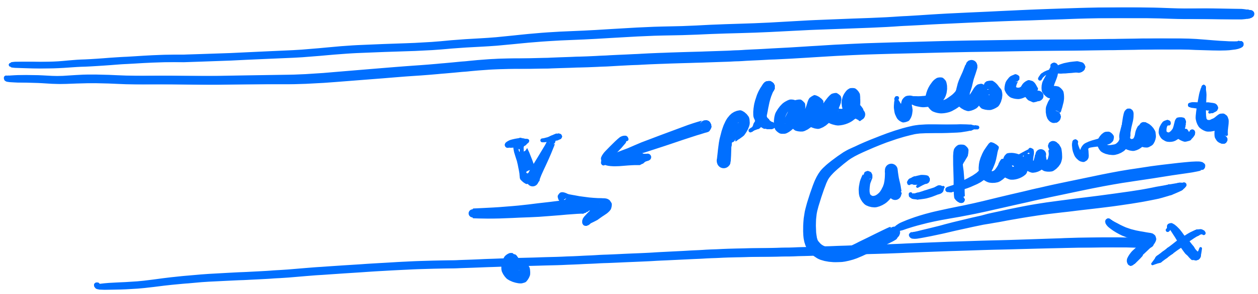
$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = y \qquad \boxed{\begin{array}{l} x = c_1 e^t \\ y = c_2 e^t \end{array}}$$



$u = F(x)$ on $y = 1$

$\Rightarrow$ gives idea on $y > 0$





Ch.

$u + c$

$u - c$

c = sound speed

$\boxed{V = u + c}$