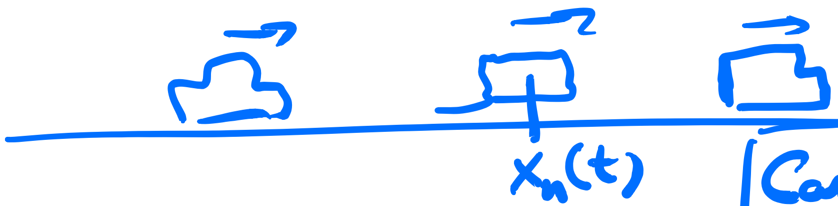
Conservation Laws and pde

Introduce densities and fluxes

Example: "Kinematic Waves"

Traffic Flow [River/channel flows]



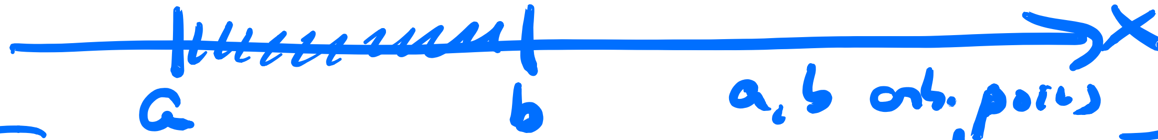
Car following theories

$\rho=\rho(x,t)$ = cars per unit length

$q=q(x,t)$ = car flux

Source $S=S(x,t)$ cars entering per unit length

Use conservation of Cars



a, b arb. points on road

$$\frac{d}{dt}\underbrace{\int_a^b \rho(x,t)\,dx}_{\#\text{cars in }[a,b]} = q(a,t) - q(b,t) + \int_a^b s(x,t)\,dx$$

Integral form of conservation law

Note a, b arbitrary

Assume ρ & q are differentiable

C^1 partials are cont

$$\int_a^b \rho_t\,dx = -\int_a^b q_x\,dx + \int_a^b s\,dx$$

$$\int_a^b [\rho_t + q_x - s]\,dx = 0$$

True for any a, b

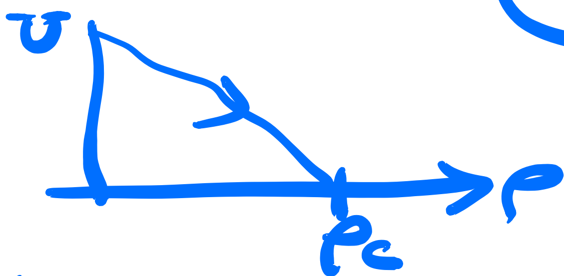
$$\Rightarrow \boxed{\rho_t + q_x = s}$$

Differential form C. Law

Now use $\boxed{\text{quasi-equilibrium assumption}}$

Car spacing $\sim \frac{1}{\rho} - L$ ↕ car length

$$u = U\left(\frac{1}{\rho} - L\right) = U(\rho)$$ equilibrium velocity



$u = U(\rho)$ $\qquad$ $q = \rho U(\rho) = Q(\rho)$

$$\rho_t + q_x = S \quad \text{and} \quad \boxed{q = Q(\rho)} \ *$$

Equation of state

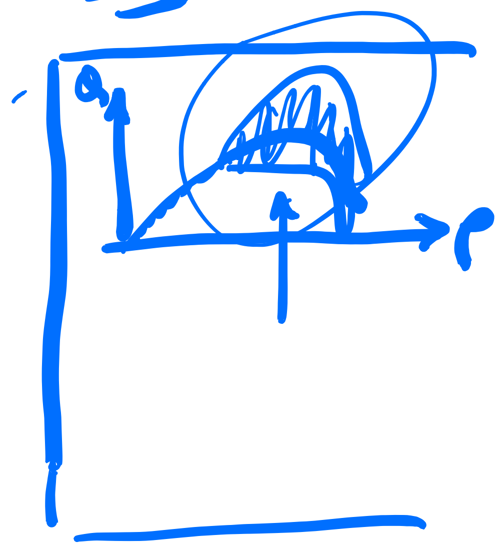
Fundamental Diagram of Traffic flow $\boxed{\text{LWR}}$

Lighthill-Whitham-Richards model 1953-54

$\boxed{Q = \rho U}$

q

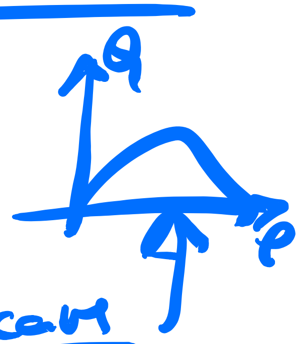
"Ideal Diagram"



LWR model

$\rho_t + q_x = S$

$q = Q(\rho)$



concave

$Q = Q(\rho)$

$Q = Q(\rho, x, t)$

Flood wave in river/channel

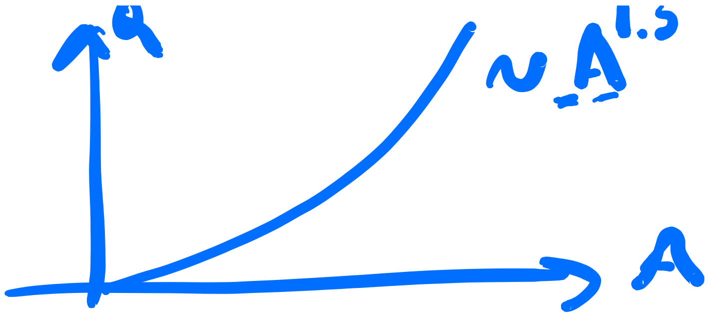
$A = A(x,t)$

$q = q(x,t)$

$\boxed{A_t + q_x = S}$

quasi-equilibrium

$\longrightarrow q = Q(A, x)$



Convex

Gas Dynamics in 1-D

INVISCID

Conserved Mass ρ = mass density

$$\boxed{\rho_t + (\rho u)_x = 0}$$

#1 C1

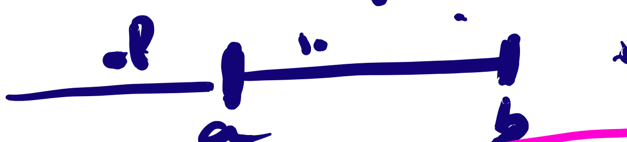
Mass flow $[\rho u]$ u = flow velocity

Cons. Mom ρu = mom. density

Mom. Flux ρu^2 ← advection

$+p$ ← Force

body force



$$\boxed{(\rho u)_t + (\rho u^2 + p)_x = F}$$

C2

Cons. Energy

$$E = \tfrac{1}{2}u^2 + e$$

Density $\frac{1}{2}\rho u^2 + \rho e = \rho E$

Flux $\underline{\rho u E + pu}$ [Ignore thermal diffusion

$$\boxed{(\rho E)_t + (\rho u E + pu)_x = F.u} \quad C_3$$

Unknown $\underline{\rho, u, p, e}$ } Need to close

Quasi-equilibrium | **Thermodynamics**

$\underline{e = e(T)}$, $\boxed{\rho, p \text{ and } T \text{ are related}}$

e.g. ideal gas laws $\quad e = c_v T$

$p = R\rho T$

$$e = \frac{c_v p}{R\rho} = \frac{p}{\gamma \rho}$$

$$\boxed{e = e(p, \rho)}$$

Equation of state

$$\underline{R = c_p - c_v}$$

$$\frac{R}{c_v} = \gamma - 1$$

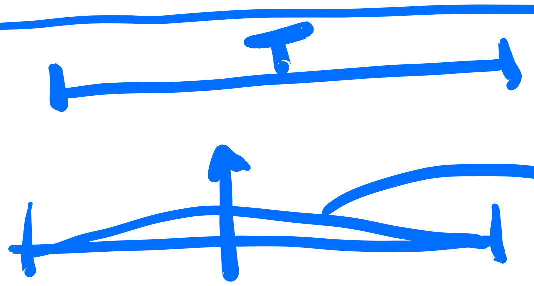
Euler Equs of Gas Dynamics

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = F$$

$$(\rho E)_t + (\rho u E + pu)_x = Fu$$

$$\boxed{e = e(p,\rho)}$$



T = tension

$u = u(x,t)$

$\boxed{\rho u_t}$ ← mom density

$$\frac{d}{dt}\int_a^b (\rho u_t)\,dx = +T u_x \Big|_a^b$$

$$= +\int_a^b (T u_x)_x\,dx$$

$$\boxed{(\rho u_t)_t - (T u_x)_x = 0}$$