18.305 Fall 2011, Solutions to HW 8

Rosalie Bélanger-Rioux
Department of Mathematics, MIT

December 5, 2011

1 Problem 1

We want to solve

$$x'' + \frac{\lambda}{1+t}x' + x = 0,\ \lambda >> 0,\ x(0) = 1,\ x'(0) = 0. \tag{1}$$

We notice first that this is not a boundary value problem but an initial value problem. And indeed, if we solve this numerically, we see that there is no such thing as a boundary layer in the solution. Instead, we use the following idea. First, let $f(t) = \frac{1}{1+t}$, and assume that f is a constant. Then we would let $x = e^{mt}$, and solve

$$m^2 + \lambda f m + 1 = 0 \Rightarrow m = \frac{-\lambda f \pm \sqrt{\lambda^2 f^2 - 4}}{2} \approx -\frac{\lambda f}{2} \pm \frac{\lambda f}{2}(1 - \frac{2}{\lambda^2 f^2})$$

or $m_1 \approx -\frac{1}{\lambda f}$, $m_2 \approx -\lambda f$. Now, since the m's vary with t, we need to write in fact $x_i = e^{\int_0^t m_i(u)du}$ or

$$x_1 = e^{-(t+t^2/2)/\lambda}$$

$$x_2 = (1+t)^{-\lambda}$$

Thus our solution is $x = Ax_1 + Bx_2$, and we solve for the constants using the initial conditions.

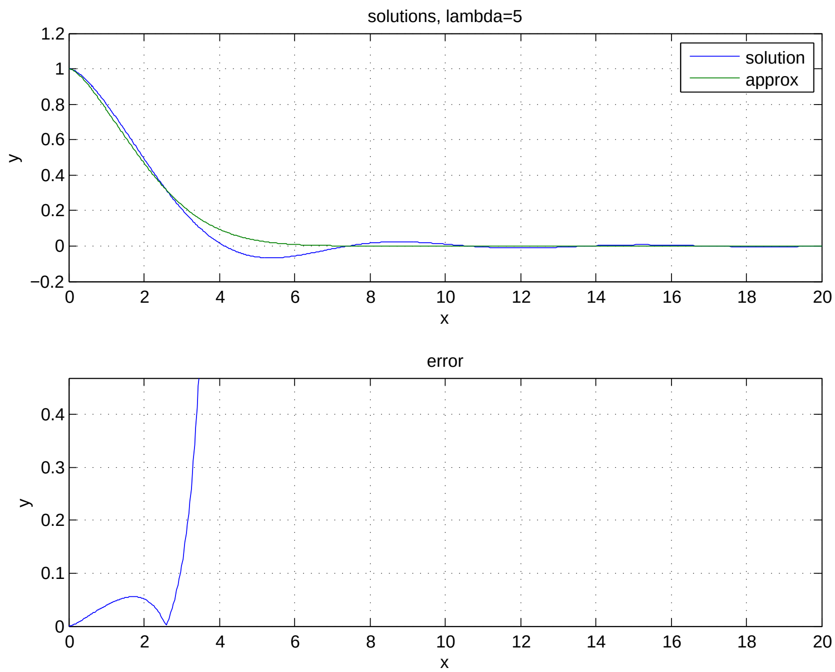
$$x(0) = A + B = 1$$

$$x'(0) = -A/\lambda - B\lambda = 0.$$

This implies that $B = \frac{1}{1-\lambda^2}$ and $A = \frac{-\lambda^2}{1-\lambda^2}$, and so the approximate solution is

$$x = \frac{-\lambda^2}{1-\lambda^2}e^{-(t+t^2/2)/\lambda} + \frac{1}{1-\lambda^2}(1+t)^{-\lambda}.$$

However, it is still not clear that this "solution" satisfies the original equation, to any order. So



we plug it in (1) to see if the ODE is satisfied:

$$
\begin{aligned}
& x'' + \frac{\lambda}{1+t}x' + x \\
= & \frac{-\lambda^2}{1-\lambda^2}x_1\left(-\frac{1}{\lambda} + \left(\frac{1+t}{\lambda}\right)^2 - 1 + 1\right) + \frac{1}{1-\lambda^2}x_2\left(\frac{\lambda}{(1+t)^2} + \left(\frac{\lambda}{1+t}\right)^2 - \left(\frac{\lambda}{1+t}\right)^2 + 1\right) \\
= & \frac{-\lambda^2}{1-\lambda^2}x_1\left(-\frac{1}{\lambda} + \left(\frac{1+t}{\lambda}\right)^2\right) + \frac{1}{1-\lambda^2}x_2\left(\frac{\lambda}{(1+t)^2} + 1\right) \\
\approx & \left(-\frac{1}{\lambda} + \left(\frac{1+t}{\lambda}\right)^2\right) + \frac{-1}{\lambda^2}\left(\frac{\lambda}{(1+t)^2} + 1\right) \\
\neq & \ 0.
\end{aligned}
$$

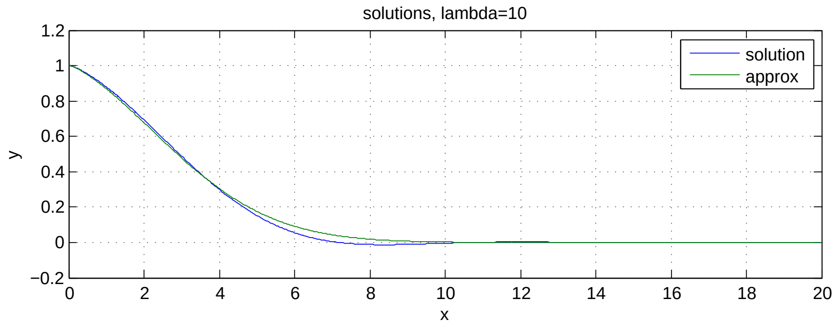
For (1) to be satisfied to order λ^{-1}, we see that we need $(1+t)^2 = O(\lambda)$ or $t^2 \leq C\lambda$.

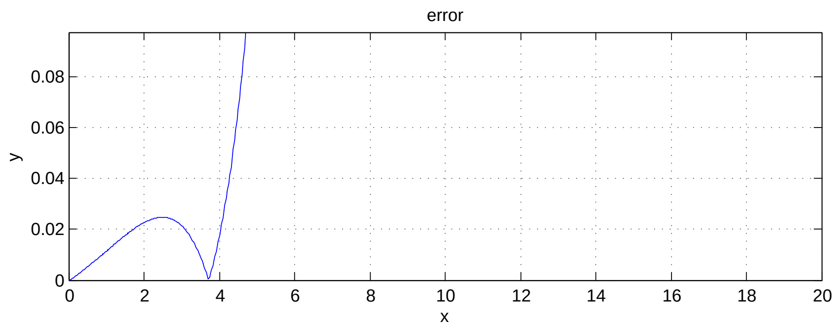
We compare it with the numerical solution and see the error decreases like λ^{-1}. We also see that the approximation, in relative error, is only good for t less than approximately the square root of λ, as expected. To get a better approximation, one would write $x = x_1 v + x_2 w$ and expand v and w in powers of λ^{-1} as we are used to, checking at every step that the approximation satisfies the equation to the desired order.

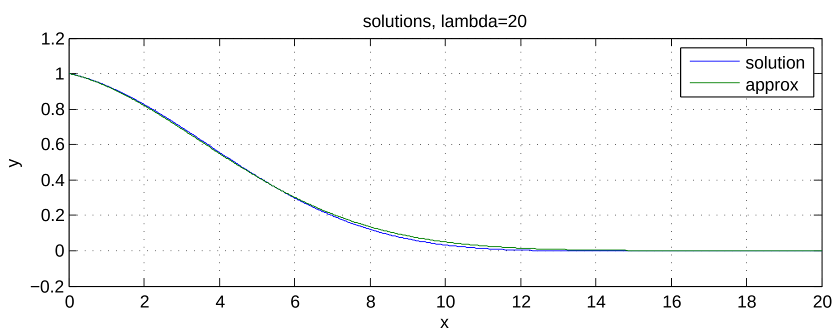
2 Problem 2

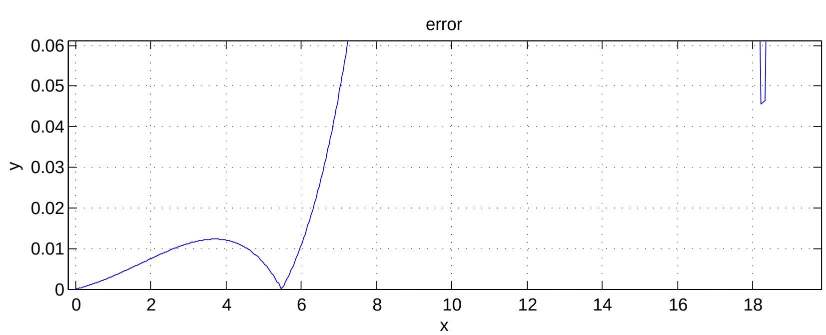
We are asked to find the solution of

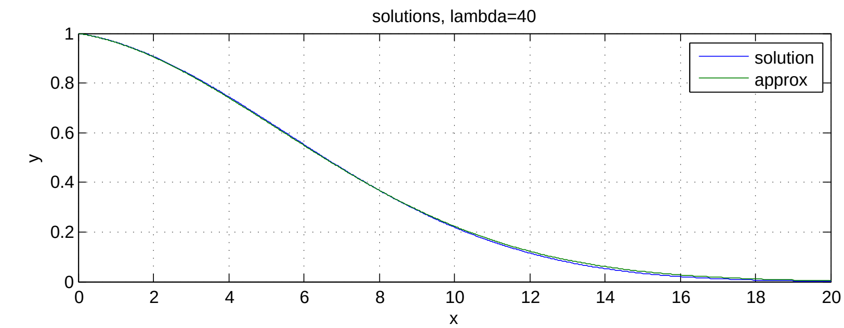
$$\epsilon y'' - (1+x)y' + y = 0, \ 0 < x < 1, \ y(0) = A, \ y(1) = B. \tag{2}$$

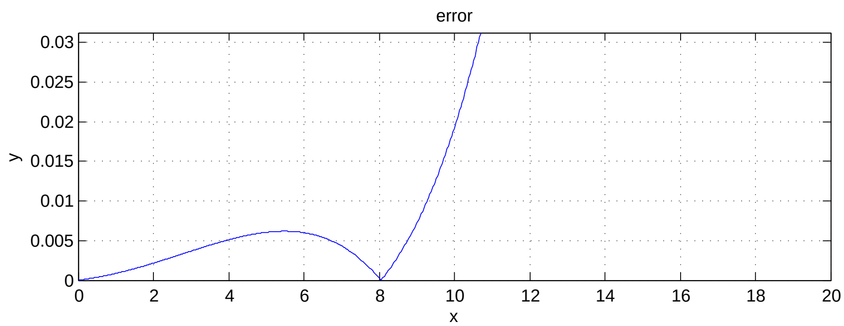

solutions, lambda=10
solution
approx
y
x


error
y
x


solutions, lambda=20
solution
approx
y
x


error
y
x





Note that $a(x) = -(1+x) < 0$ so the boundary layer is at $x = 1$. We first find the slowly varying solution :

$$y^s = y_0^s + \epsilon y_1^s + ...$$

Putting this in (2) gives to zeroth order

$$y_0'^s = \frac{y_0^s}{1+x} \Rightarrow y_0^s = C(1+x).$$

But we know the boundary layer is at $x = 1$, so the rapidly varying solution is exponentially small near $x = 0$, which means that we need $y^s(0) = A = y_0^s(0)$ and $y_1^s(0) = 0$. Hence we have $y_0^s = A(1+x)$. Now, to first order in ϵ we get

$$-(1+x)y_1'^s + y_1^s = -y_0''^s = 0 \Rightarrow y_1^s = D(1+x) = 0$$

since D must be 0 to satisfy the boundary condition at $x = 0$. So the slowly varying solution is (note that in fact $y_n^s = 0 \ \forall \ n \geq 1$, but you were not asked to show this):

$$y^s = A(1+x).$$

Now we look for the rapidly varying solution. As in the book, we assume

$$y^r = e^{\int_x^1 \frac{a(t)}{\epsilon} dt} v(x), \ \ v = v_0 + \epsilon v_1 + ...$$

Note that some students integrated the function a from 0 to x instead of from x to 1 as above. This works too, it just makes things a little bit more complicated. Now, again, the book gives the relevant ode we obtain for v:

$$av' + (a' - b)v = \epsilon v''$$

where b is the coefficient in front of y in (2), hence $b = 1$. Also, $a' = -1$, so we get to zeroth order

$$-(1+x)v_0' - 2v_0 = 0 \Rightarrow v_0 = \frac{E}{(1+x)^2}$$

where we need to find E from the boundary condition. We know that $y = y^s + y^r$, hence at $x = 1$ we get $B = 2A + v_0(1)$ and $v_1(0) = 0$. This means that $E = 4(B - 2A)$. Now, we find v to first order:

$$-(1+x)v_1' - 2v_1 = v_0'' = \frac{24(B-2A)}{(1+x)^4}.$$

We know how to solve this using an integrating factor. The solution is:

$$\begin{aligned} v_1 &= e^{-\int 2/(1+t)dt} \int e^{\int 2/(1+t)dt} \frac{-24(B-2A)}{(1+x)^5} dx \\ &= (1+x)^{-2} \int \frac{-24(B-2A)}{(1+x)^3} dx \\ &= (1+x)^{-2} \left(\frac{12(B-2A)}{(1+x)^2} + F \right) \end{aligned}$$

where we use the boundary condition $v_1(1) = 0$ to determine that $F = -3(B - 2A)$. Finally, we note that $\int_x^1 \frac{a(t)}{\epsilon} dt = -\frac{1-x+1/2-x^2/2}{\epsilon}$. Hence we have found the solution of (2) up to and including the first order:

$$y = A(1+x) + e^{\frac{x+x^2/2-3/2}{\epsilon}} \left(\frac{B-2A}{(1+x)^2} \right) \left(4 + 3\epsilon \left(\frac{4}{(1+x)^2} - 1 \right) \right).$$

We solve this numerically and obtain good agreement.