18.305 Fall 2011, Solutions to HW 6

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1 Problem 1

We are asked to solve $y'' + xy' + (n+1)y = 0$ for n a non-negative integer.

(a) Using the Laplace integral representation, let $y(x) = \int_C e^{-xt} Y(t) dt$ where C is a contour to be determined. Then $y' = \int_C -te^{-xt} Y(t) dt$, $y'' = \int_C t^2 e^{-xt} Y(t) dt$, $xy' = -\int_C e^{-xt} \frac{d}{dt}(tY(t))\, dt$ provided that

$$e^{-xt} tY(t)\Big|_{t=a}^{t=b} = 0 \tag{1}$$

where a and b are the endpoints of the contour, possibly infinite. The equation then becomes $t^2 Y - (t\frac{dY}{dt} + Y) + (n+1)Y = 0$. One can easily solve this equation (details left to the reader) to obtain $Y(t) = t^n e^{t^2/2}$. Now, let $t = re^{i\theta}$, then $|Y(t)| = r^n e^{r^2 \cos(2\theta)/2}$ so that the regions where this Y will blow up are for $-\pi/4 \leq \theta \leq \pi/4$ and $3\pi/4 \leq \theta \leq 5\pi/4$. Now, recall we need to satisfy the condition (**??**) for the endpoints. This is clearly satisfied when $t = 0$ and also when $t = re^{i\theta} \to \infty$ with θ in one of the allowed sectors. We may thus choose two independent paths of integration, say C_1 starting from 0 and going up the positive imaginary axis, and C_2 starting from zero again but going down the negative imaginary axis. Thus we get two independent solutions: $y_1 = \int_{C_1} e^{-xt} t^n e^{t^2/2} dt$ and $y_2 = \int_{C_2} e^{-xt} t^n e^{t^2/2} dt$.

(b) Now we want to find the asymptotics of these solutions for large $|x|$. Notice that, on the chosen contours, we have $t = \pm i\tau$ for real positive τ, hnce

$$y_{1,2} = \int_0^\infty e^{\mp ix\tau} (\pm i\tau)^n e^{-\tau^2/2} d\tau.$$

so that the contributions coming from the infinite endpoints for each solution y_1 and y_2 will be exponentially small. However, the contributions coming from the $\tau = 0$ endpoint will not. Those in fact will be the largest, so we need compute them. Hence, considering now the endpoint $\tau = 0$, we may drop the factor $e^{-\tau^2/2} \approx 1$ to obtain

$$y_{1,2} \approx \int_0^\infty e^{\mp ix\tau} (\pm i\tau)^n d\tau.$$

We have seen this integral often before; just deform the contour again ($\tau = \mp is$, s real positive) and get (omitting constant factors)

$$y_{1,2} \approx \int_0^\infty e^{-xs} s^n ds = \frac{n!}{x^{n+1}}$$

for positive x and ($\tau = \pm is$, s real positive, omitting constant factors)

$$y_{1,2} \approx \int_0^\infty e^{xs}(-s)^n ds = \frac{n!}{x^{n+1}}$$

for negative x.

2 Problem 2

$$I = \int_{-\infty}^{\infty} e^{ix^4} dx = 2\int_0^\infty e^{ix^4} dx$$

Since this integrand is analytic, we may change the contour of integration. For this, we want to write $z^4 = it^4$, and we need to choose the appropriate root: $z = e^{i\pi/8}t$. We shall thus integrate over the contour C made of the straigth line at an angle of $\pi/8$ until we hit the circle of radius R around the origin, at which point we go straight back down to the real axis. We then need to take the limit $R \to \infty$. Thus we have:

$$I = 2\lim_{R\to\infty}\int_C e^{ix^4} dx = 2\lim_{R\to\infty}\left(e^{i\pi/8}\int_0^R e^{-t^4} dt + \int_0^{Rsin(\pi/8)} e^{i(R+iy)^4} idy\right).$$

Note that the size of the second term is bounded by the integral of e to the real part of the power, $\mathcal{R}(i(R+iy)^4) = i(4iyR^3 + 6iyR)$. Hence the second term goes to 0 as we take the limit $R \to \infty$. And we easily recognize the first term as a Gamma function. Hence

$$I = 2e^{i\pi/8}\lim_{R\to\infty}\int_0^R e^{-t^4} dt = \frac{2e^{i\pi/8}\Gamma(1/4)}{4} = \frac{e^{i\pi/8}\Gamma(1/4)}{2}.$$

3 Problem 3

$$I = \int_0^\infty e^{i\lambda z - 1/z} dz = \lambda^{-1/2}\int_0^\infty e^{i\Lambda(t+i/t)} dt = \Lambda^{-1}\int_0^\infty e^{i\Lambda f(t)} dt$$

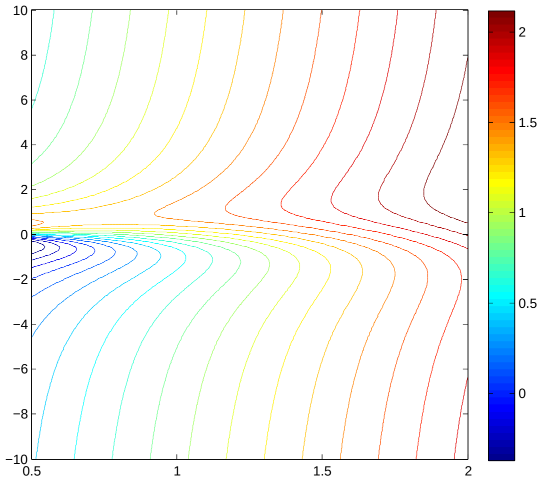
with the change of variables $\lambda^{1/2}z = t$, $f(t) = t + i/t$, $\Lambda = \lambda^{1/2}$. Notice there are no points of stationary phase, and the contributions from the endpoints are exponentially small. Hence we find the saddle points:

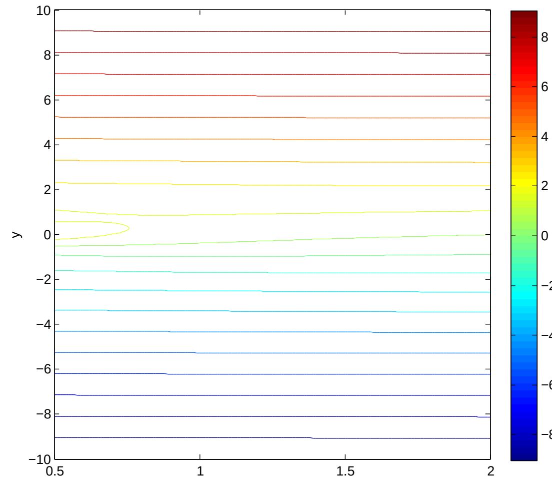
$$f'(t_0) = 1 - i/t_0^2 = 0 \Leftrightarrow t_0^2 = i \Leftrightarrow t_0 = \pm e^{i\pi/4} = \frac{\pm(1+i)}{\sqrt{2}},$$

with $f''(t) = 2i/t^3$. Note that $f(t_0) = \frac{\pm 1}{\sqrt{2}}(1+i) \pm i\sqrt{2}(1-i)/2 = \pm\sqrt{2}(1+i)$. Since it is obvious that I is exponentially small (because the integrand oscillates very fast), we want to deform the contour onto the saddle point on which the integrand $e^{i\Lambda f(t_0)}$ is small – thus we need to choose the plus sign:

$$t_0 = e^{i\pi/4} = \frac{(1+i)}{\sqrt{2}}.$$

Now we want to deform the path of integration along the path of steepest descent through this saddle point. However, this looks difficult here because it is not clear what the steepest paths are. But we do not really need them to find the largest contribution to the integral - all we want is a path of descent. This means we want to make sure there is a path through the saddle point along





which $u(x,y)$, the phase of f, is constant while $v(x,y)$ attains a minimum at the saddle point. Why? Remember: because we have

$$I = \Lambda^{-1}\int_0^\infty e^{i\Lambda f(t)}dt = \Lambda^{-1}\int_0^\infty e^{i\Lambda(u(x,y)+iv(x,y))}dt = \Lambda^{-1}\int_0^\infty e^{i\Lambda u(x,y)}e^{-\Lambda v(x,y)}dt$$

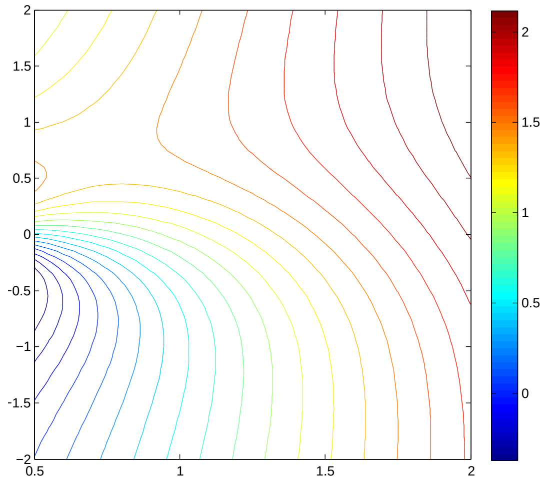
with $t = x+iy$. One may easily compute $u(x,y) = x + \frac{y}{x^2+y^2}$ and $v(x,y) = y + \frac{x}{x^2+y^2}$. We will deform the contour of integration in the upper half plane, through the chosen saddle point and along the level curve of u which gives a path of steepest descent for the integrand (so where v has a minimum). If we plot the level curves we may see that, if we follow a level curve of u through the saddle point we indeed reach a minimum of v at the saddle point. So, although we do not know the exact path along which we should be integrating, we can evaluate the main contribution of the integral I, the contribution from points close to the saddle point. (See figures, where the left images show the level curves of u, the right ones the level curves of v: clearly one level curve of u corresponds to a path of steepest ascent, the other to a path of steepest descent, and we have chosen the correct saddle point.) And why can we deform the contour to the upper half plane? We may write $v = r\sin\theta + \cos\theta/r$ for $t = re^{i\theta}$, and it is clear that as $r \to \infty$, we need to be in the upper half plane so that v remains positive and increasing.

Now that we have justified the method (this is what part b asked for), we may write down the main contribution, noting that $f''(t_0) = 2i/t_0^3 = 2e^{-i\pi/4}$.

$$I(\lambda) \approx \Lambda^{-1}\int_0^\infty e^{i\Lambda\left(f(t_0)+f''(t_0)(t-t_0)^2/2\right)}dt = \Lambda^{-1}e^{i\Lambda f(t_0)}\sqrt{\frac{2i\pi}{\Lambda f''(t_0)}}$$

or finally

$$I(\lambda) \approx \lambda^{-3/4}e^{\sqrt{2\lambda}(i-1)}\sqrt{\pi}e^{i3\pi/8}.$$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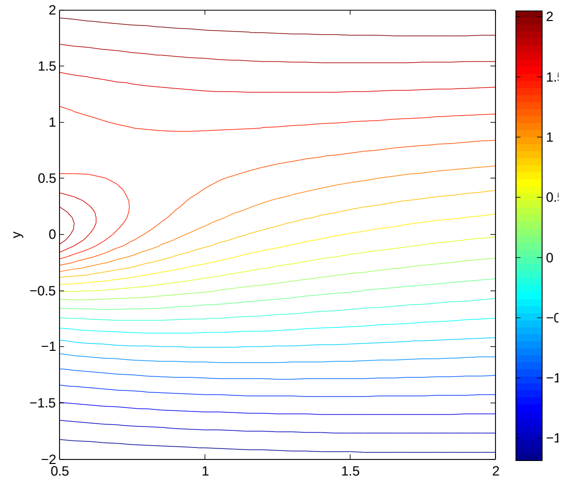
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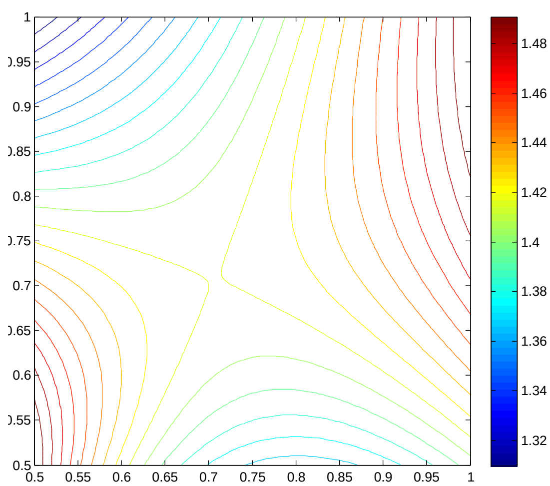
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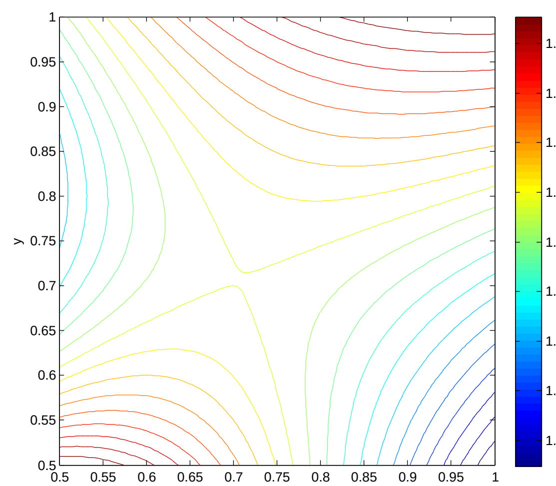
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