18.305 Fall 2011, Solutions to HW 4

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1 Chapter 8, Problem 3

1.1 Solutions to part (a)

$$I(\lambda) = \int_{-1}^{1} e^{-\lambda t^3} dt$$

We have

$$v(t) = t^3,$$

and

$$h(t) = 1$$

$v(t)$ is minimum at the lower endpoint $t = -1$, and $v(-1) = -1$, $v'(-1) = 3$. Thus, we have

$$\boxed{I(\lambda) \approx \frac{e^{\lambda}}{3\lambda}} \tag{1}$$

1.2 Solutions to part (b)

$$I(\lambda) = \int_{1}^{\infty} e^{-\lambda t^2} dt$$

We have

$$v(t) = t^2,$$

and

$$h(t) = 1$$

$v(t)$ is minimum at the lower endpoint $t = 1$, and $v(1) = 1$, $v'(1) = 2$. Thus, we have

$$\boxed{I(\lambda) \approx \frac{e^{-\lambda}}{2\lambda}} \tag{2}$$

1.3 Solutions to part (c)

$$\begin{aligned} I(\lambda) &= \int_{-2}^{1} \sin x e^{-\lambda x^2} dx \\ &= \int_{-2}^{-1} \sin x e^{-\lambda x^2} dx + \int_{-1}^{1} \sin x e^{-\lambda x^2} dx \\ &= \int_{-2}^{-1} \sin x e^{-\lambda x^2} dx \end{aligned}$$

The integral from -1 to 1 vanishes because the integrand is an odd function. We then have

$$v(x) = x^2,$$

and

$$h(x) = \sin x$$

$v(x)$ is minimum at the upper endpoint $x = -1$, and $v(-1) = 1$, $v'(-1) = 2x$, $h(x) = \sin(-1)$. Thus, we have

$$\boxed{I(\lambda) \approx -\frac{e^{-\lambda} \sin(-1)}{-2\lambda} = -\frac{e^{-\lambda}}{2\lambda} \sin 1} \tag{3}$$

1.4 Solutions to part (d)

$$I(\lambda) = \int_{-\pi}^{\pi} e^{-\lambda \sin x} dx$$

We have

$$v(x) = \sin x,$$

and

$$h(x) = 1$$

$v(x)$ is minimum at an interior point $x = -\pi/2$, and $v(-\pi/2) = -1$, $v''(-\pi/2) = 1$. Thus, we have

$$\boxed{I(\lambda) \approx e^{\lambda} \sqrt{\frac{2\pi}{\lambda}}} \tag{4}$$

1.5 Solutions to part (e)

$$I(\lambda) = \int_{0}^{\infty} e^{-\lambda x} e^{-x^2} dx$$

We have

$$v(x) = x,$$

and

$$h(x) = e^{-x^2}$$

$v(x)$ is minimum at the lower endpoint $x = 0$, and $v(0) = 0$, $v'(0) = 1$, $h(0) = 1$. Thus, we have

$$\boxed{I(\lambda) \approx \frac{1}{\lambda}} \tag{5}$$

1.6 Solutions to part (f)

$$I(\lambda) = \int_0^\lambda e^{t^3}\,dt$$

Applying the change of variable

$$t = \lambda x,$$

the integral becomes

$$I(\lambda) = \lambda \int_0^1 e^{\Lambda x^3}\,dx,$$

where

$$\Lambda = \lambda^3$$

We see that the dominant contribution comes from a small neighborhood near $x = 1$. Expanding around that point,

$$x^3 \approx 1 + 3(x-1)$$

Thus,

$$\begin{aligned}
I(\lambda) &= \lambda \int_0^1 e^{\Lambda x^3}\,dx \\
&\approx \lambda e^\Lambda \int_0^1 e^{3\Lambda(x-1)}\,dx \\
&= \lambda e^\Lambda \left[\frac{e^{3\Lambda(x-1)}}{3\Lambda}\right]_0^1 \\
&= \lambda e^\Lambda \left[\frac{1-e^{-3\Lambda}}{3\Lambda}\right] \\
&\approx \lambda \frac{e^\Lambda}{3\Lambda}
\end{aligned}$$

$$\boxed{I(\lambda) \approx \frac{e^{\lambda^3}}{3\lambda^2}} \tag{6}$$

1.7 Solutions to part (g)

$$I(\lambda) = \int_0^\infty e^{-\lambda(t+t^5)}\,dt$$

We have

$$v(t) = t + t^5,$$

and

$$h(t) = 1$$

$v(t)$ is minimum at the lower endpoint $t = 0$, and $v(0) = 0$, $v'(0) = 1$. Thus, we have

$$\boxed{I(\lambda) \approx \frac{1}{\lambda}} \tag{7}$$

2 Chapter 8, Problem 5

$$I(\lambda) = \int_0^\infty e^{-\lambda x - \frac{1}{x}} dx$$

As was pointed out in the hint, we have to take into account both factors. Let

$$h(x) = x + \frac{1}{\lambda x}.$$

The minimum of $h(x)$ occurs at

$$\frac{1}{\sqrt{\lambda}}.$$

To simplify our calculations, we apply the change of variables

$$x = \frac{1}{\sqrt{\lambda}} t.$$

Thus,

$$\begin{aligned} I(\lambda) &= \int_0^\infty e^{-\lambda x - \frac{1}{x}} dx \\ &= \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}\left(t + \frac{1}{t}\right)} dt \end{aligned}$$

Now $h(t) = t + 1/t$ and $v(t) = 1$. The minimum of $h(t)$ occurs at $t = 1$. Around $t = 1$, we can make the approximation

$$h(t) = 2 + \frac{h''(t)}{2} (t-1)^2 = 2 + (t-1)^2.$$

Therefore,

$$\begin{aligned} I(\lambda) &= \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}\left(2 + (t-1)^2\right)} dt \\ &= \frac{e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}(t-1)^2} dt \\ &= \frac{e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}} \sqrt{\frac{\pi}{\sqrt{\lambda}}} \\ &= \frac{e^{-2\sqrt{\lambda}}\sqrt{\pi}}{\lambda^{3/4}} \end{aligned}$$

3 Chapter 8, Problem 7

3.1 Solutions to part (a)

$$I(\lambda) = \int_{-1}^1 e^{-\lambda t^3} dt$$

Since the dominant contribution to this integral is around the endpoint -1, we may change the upper limit from 1 to ∞.

$$\begin{aligned}
I(\lambda) = \int_{-1}^{1} e^{-\lambda t^3} dt &\approx \int_{-1}^{\infty} e^{-\lambda t^3} dt \\
&\overset{s=t^3}{=} \int_{-1}^{\infty} e^{-\lambda s} \frac{1}{3s^{2/3}} ds \\
&\overset{x=s+1}{=} \int_{0}^{\infty} e^{-\lambda(x-1)} \frac{1}{3(x-1)^{2/3}} dx \\
&= \frac{e^{\lambda}}{3} \int_{0}^{\infty} e^{-\lambda x} \frac{1}{(1-x)^{2/3}} dx \\
&\overset{y=\lambda x}{=} \frac{e^{\lambda}}{3\lambda} \int_{0}^{\infty} e^{-y} (1-\frac{y}{\lambda})^{-2/3} dy \\
&= \frac{e^{\lambda}}{3\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)}{\Gamma(2/3) n! \lambda^n} \int_{0}^{\infty} e^{-y} y^n dy \\
&= \frac{e^{\lambda}}{3\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)\Gamma(n+1)}{\Gamma(2/3) n! \lambda^n} \\
&= \frac{e^{\lambda}}{3\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)}{\Gamma(2/3)\lambda^n}
\end{aligned}$$

3.2 Solutions to part (b)

$$\begin{aligned}
I(\lambda) &= \int_{1}^{\infty} e^{-\lambda t^2} dt \\
&\overset{s=t^2}{=} \int_{1}^{\infty} e^{-\lambda s} \frac{1}{2\sqrt{s}} ds \\
&\overset{x=s-1}{=} \int_{0}^{\infty} e^{-\lambda(x+1)} \frac{1}{2(x+1)^{1/2}} dx \\
&= \frac{e^{-\lambda}}{2} \int_{0}^{\infty} e^{-\lambda x} (1+x)^{-1/2} dx \\
&\overset{y=\lambda x}{=} \frac{e^{-\lambda}}{2\lambda} \int_{0}^{\infty} e^{-y} (1+\frac{y}{\lambda})^{-1/2} dy \\
&= \frac{e^{-\lambda}}{2\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)(-1)^n}{\Gamma(1/2) n! \lambda^n} \int_{0}^{\infty} e^{-y} y^n dy \\
&= \frac{e^{-\lambda}}{2\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(1/2)(-\lambda)^n}
\end{aligned}$$

3.3 Solutions to part (d)

$$I(\lambda) = \int_{-\pi}^{\pi} e^{-\lambda \sin x} dx$$

Since the dominant contribution to the integral is around $x = -\pi/2$, we have

$$\begin{aligned}
I(\lambda) &\approx 2\int_{-\pi/2}^{\pi/2} e^{-\lambda \sin x} dx \\
&\overset{t=\sin x}{=} 2\int_{-1}^{1} e^{-\lambda t}\frac{1}{\sqrt{1-t^2}} dt \\
&\overset{s=t+1}{=} 2\int_{0}^{2} e^{-\lambda (s-1)}\frac{1}{(1-(s-1)^2)^{1/2}} ds \\
&\approx 2e^{\lambda}\int_{0}^{\infty} e^{-\lambda s}\frac{1}{\sqrt{2s}\sqrt{1-s/2}} ds \\
&\overset{y=\lambda s}{=} \frac{\sqrt{2}e^{\lambda}}{\sqrt{\lambda}}\int_{0}^{\infty} e^{-y}\frac{1}{\sqrt{y}}(1-\frac{y}{2\lambda})^{-1/2} dy \\
&= \frac{\sqrt{2}e^{\lambda}}{\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\Gamma(1/2)2^n\lambda^n n!}\int_{0}^{\infty} e^{-y}y^{(n-1/2)} dy \\
&= \frac{\sqrt{2}e^{\lambda}}{\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+1/2)}{\Gamma(1/2)2^n\lambda^n n!}
\end{aligned}$$

3.4 Solutions to part (e)

$$\begin{aligned}
I(\lambda) &= \int_{0}^{\infty} e^{-\lambda x}e^{-x^2} dx \\
&\overset{y=\lambda x}{=} \frac{1}{\lambda}\int_{0}^{\infty} e^{-y}e^{-(y/\lambda)^2} dy \\
&= \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(-\lambda)^{2n}}\int_{0}^{\infty} e^{-y}y^{2n} dy \\
&= \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2n+1)}{n!\lambda^{2n}} \\
&= \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{n!\lambda^{2n}}
\end{aligned}$$

3.5 Solutions to part (f)

$$\begin{aligned}
I(\lambda) &= \int_0^\lambda e^{t^3} dt \\
&\overset{t^3=x+\lambda^3}{=} \int_{-\lambda}^0 \frac{e^{x+\lambda^3}}{3(x+\lambda^3)^{2/3}} dx \\
&= \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^0 \frac{e^x}{(1+x/\lambda^3)^{2/3}} dx \\
&\approx \frac{e^{\lambda^3}}{3\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)(-1)^n}{\Gamma(2/3)\lambda^{3n} n!} \int_{-\infty}^0 e^x x^n dx \\
&\approx \frac{e^{\lambda^3}}{3\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)}{\Gamma(2/3)\lambda^{3n}}
\end{aligned}$$

3.6 Solutions to part (g)

$$\begin{aligned}
I(\lambda) &= \int_0^\infty e^{-\lambda(t+t^5)} dt \\
&\overset{x=\lambda t}{=} \int_0^\infty \frac{e^{-x} e^{-\lambda(x/\lambda)^5}}{\lambda} dx \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \lambda^{4n}} \int_0^\infty e^{-x} x^{5n} dx \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n (5n)!}{n! \lambda^{4n}}
\end{aligned}$$