18.305 Fall 2011, Solutions to HW 1

Rosalie Bélanger-Rioux
Department of Mathematics, MIT

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1 Chapter 1, Problem 1(a)

First note that when $x = 0, 1$ the equation is not first order anymore. So assume $x \neq 0, 1$ and divide to get

$$y' + \frac{1}{2x} y = \frac{1}{\sqrt{x}(1+x)}.$$

We may solve this using an integrating factor of $\exp\left(\int \frac{1}{2x} dx\right) = \sqrt{x}$, to obtain

$$\frac{d}{dx}\left(\sqrt{x} y\right) = \frac{1}{1+x}$$

$$\Rightarrow y = \frac{1}{\sqrt{x}}\left(c + \ln(1+x)\right)$$

is the solution for some arbitrary c.

2 Chapter 1, Problem 1(c)

Note that $y \equiv 0$ is a solution. For $y \neq 0$, let $u = 1/y$, $u' = \frac{-y'}{y^2} = -u^2 y'$, so that the ODE becomes $\frac{-u'}{u^2} = \frac{1}{xu} + xu^2$ or $u' + \frac{u}{x} = -x$. We use an integrating factor of $\exp\left(\int \frac{1}{x} dx\right) = x$ to get

$$\frac{d}{dx}(xu) = -x^2 \Rightarrow xu = \frac{-x^3}{3} + c \Rightarrow u = \frac{1}{x}\left(\frac{-x^3}{3} + c\right)$$

$$\Rightarrow y = \frac{x}{c - x^3/3}$$

is the solution for some arbitrary c.

3 Chapter 2, Problem 2(b)

We first look for the homogeneous solution, which solves $y'' + y' + y = 0$. The characteristic polynomial is $m^2 + m + 1 = 0$, with roots $m_\pm = .5\left(-1 \pm \sqrt{3}i\right)$. Thus the homogeneous solution is $c_1 e^{-x/2} \cos \sqrt{3}x/2 + c_2 e^{-x/2} \sin \sqrt{3}x/2$ for arbitrary c_i's. And we find a particular solution by solving

$$(D^2 + D + 1)\tilde{y} = e^{ix}$$

where we note that $D \to i$ in this particular case, so that we obtain $i\tilde{y} = e^{ix}$, where the particular solution to our original problem is

$$y_p = \Re\left[y_p\right] = \Re\left[\frac{e^{ix}}{i}\right] = \sin x.$$

So the solution to the original problem is

$$y = \sin x + e^{-x/2}\left(c_1 \cos \sqrt{3}x/2 + c_2 \sin \sqrt{3}x/2\right).$$

4 Chapter 2, Problem 2(d)

We wish to solve the equidimensional equation

$$x^3 y''' + 3x^2 y'' + 3xy' + y = \ln x$$

by the change of variable $x = e^t$. From the book, we know that $x^n \frac{d^n}{dx^n} = D(D-1)\cdots(D-n+1)$ where now $D = \frac{d}{dt}$. So the equation becomes

$$(D(D-1)(D-2) + 3D(D-1) + 3D + 1)\, y = t \Rightarrow \left(D^3 + 2D + 1\right) y = t.$$

We solve the homogeneous equation $\left(D^3 + 2D + 1\right) y = 0$, with a characteristic polynomial of $m^3 + 2m + 1 = 0$ which has roots m_1, m_2 and m_3, to obtain the homogeneous solution $c_1 e^{m_1 t} + c_2 e^{m_2 t} + c_3 e^{m_3 t} = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3}$ for some arbitrary c_i's. To find a particular solution, we solve

$$y_p = \frac{1}{D^3 + 2D + 1} t = \frac{1}{1 - (-2D - D^3)} t = (1 - 2D - D^3 + \ldots)t$$

where we have Taylor expanded. Since $D^n t = 0$ for $n > 1$, this gives $y_p = (1-2D)t = t - 2 = \ln x - 2$. We have obtained the solution to the original equation:

$$y = \ln x - 2 + c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3}.$$

5 Chapter 2, Problem 2(f)

We wish to solve

$$y''' + y'' - y' - y = xe^x + 7\cosh x.$$

The homogeneous equation has characteristic polynomial $m^3 + m^2 - m - 1 = (m-1)(m+1)^2 = 0$ so that the homogeneous solution is $c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$ for arbitrary c_i's. Now we wish to find a particular solution. For the first term in the right-hand-side, we solve

$$y_{p_1} = \frac{1}{(D+1)^2(D-1)} xe^x = e^x \frac{1}{(D+2)^2 D} x = e^x \frac{1}{4 + 4D + D^2} \frac{x^2}{2} = \frac{e^x}{8} \frac{1}{1 + D + D^2/4} x^2.$$

We may Taylor expand to get

$$y_{p_1} = \frac{e^x}{8}\left(1 - D - D^2/4 + D^2 + \ldots\right) x^2 = \frac{e^x}{8}\left(x^2 - 2x + 3/2\right),$$

where the last term is not important since it can be absorbed into the homogeneous solution. The procedure is similar for the other right-hand-side, which we write as $7\cosh x = 7(e^x + e^{-x})/2$.

$$y_{p_2} = \frac{7}{2}\frac{1}{(D+1)^2(D-1)}e^x = e^x \frac{7}{2}\frac{1}{D(D+2)^2}1 = e^x \frac{7}{8}\frac{1}{D}1 = \frac{7xe^x}{8},$$

using twice the fact that $\frac{1}{D+2}1 = 1/2$. And finally, using $\frac{1}{D-2}1 = -1/2$:

$$y_{p_3} = \frac{7}{2}\frac{1}{(D+1)^2(D-1)}e^{-x} = e^{-x}\frac{7}{2}\frac{1}{D^2(D-2)}1 = e^{-x}\frac{-7}{4}\frac{1}{D^2}1 = \frac{-7x^2e^{-x}}{8}.$$

Thus the solution to the original problem is

$$y = c_1e^x + c_2e^{-x} + c_3xe^{-x} + \frac{x^2e^x}{8} + \frac{5xe^x}{8} - \frac{7x^2e^{-x}}{8}.$$