Answers to Problem Set Number 1 for 18.305 — MIT (Fall 2005)

D. Margetis and R. Rosales (MIT, Math. Dept., Cambridge, MA 02139).

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Course TA: Nikos Savva, MIT, Dept. of Mathematics, Cambridge, MA 02139.

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1 Bender and Orszag Problem 3.4 (d) (e) (f)

Classify the points at 0 and ∞ of the following differential equations:

(d) $x^2 y'' = \mathrm{e}^{1/x} y$.

(e) $(\tan x)\, y' = y$.

(f) $y'' = y \ln x$.

Solution Recall that for the differential equations of the form

$$y'' + p_1(x)\, y' + p_0(x)\, y = 0, \tag{1.1}$$

a point x_0 is called

- *Ordinary:* if and only if both p_1 and p_0 are analytic in a neighborhood of x_0.
- *Regular singular:* if and only if either of p_1 or p_0 fails to be analytic at x_0, but both $(x - x_0)\, p_1(x)$ and $(x - x_0)^2\, p_0(x)$ are analytic in a neighborhood of x_0.
- *Irregular singular:* if and only if it is neither an ordinary nor a regular singular point.

(d) For this differential equation, $x = 0$ is an irregular singular point because $\mathrm{e}^{1/x}/x^2$ has an essential singularity at $x = 0$.

Using the transformation $t = 1/x$ we get

$$t^2 \ddot{y} + 2t\dot{y} = e^t y \qquad \Rightarrow \qquad \ddot{y} + \frac{2}{t}\dot{y} - \frac{e^t}{t} y = 0, \tag{1.2}$$

from which we conclude that $x = \infty$ is a regular singular point.

(e) Here we have

$$y' = \frac{y}{\tan x} = y\left(\frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 + \mathcal{O}\left(x^5\right)\right). \tag{1.3}$$

Hence $x = 0$ is a regular singular point. The transformation $t = 1/x$ gives

$$\frac{dy}{dt} = -\frac{y}{t^2 \tan(1/t)}, \tag{1.4}$$

which has an essential singularity at $t = 0$. Hence $x = \infty$ is an irregular singular point.

(f) We can immediately see that $x = 0$ is an irregular singular point, because of the presence of the logarithm (with a branch point there). The same applies for ∞, because under the transformation $t = 1/x$ the original differential equation becomes

$$\ddot{y} + \frac{2}{t}\dot{y} + y\frac{\ln t}{t^4} = 0, \tag{1.5}$$

which also has 0 as an irregular singular point.

2 Bender and Orszag Problem 3.6

Find the Taylor series valid near $x = 0$ for the solution to the initial-value problem

$$y'' - 2xy' + 8y = 0, \quad \text{with} \quad y(0) = 4 \quad \text{and} \quad y'(0) = 0. \tag{2.1}$$

Note: write ALL the coefficients in the Taylor expansion explicitly.

Solution The Taylor series valid near $x = 0$ has the form

$$y = \sum_{n=0}^{\infty} a_n x^n, \tag{2.2}$$

where $a_0 = 4$ and $a_1 = 0$, as required by the initial conditions. Substituting this series in the equation and equating equal powers of x yields

$$(n+2)(n+1)\, a_{n+2} - 2(n-4)\, a_n = 0. \tag{2.3}$$

Since $a_1 = 0$, we must have $a_n = 0$ for n odd. For even $n = 2m - 2$ $(m \geq 1)$ we obtain:

$$a_{2m} = \frac{2(m-3)}{m(2m-1)}\, a_{2(m-1)}. \tag{2.4}$$

Hence

$$a_2 = -16, \quad a_4 = \frac{16}{3}, \quad \text{and} \quad a_{2m} = 0 \quad \text{for} \quad m \geq 3. \tag{2.5}$$

The solution is thus

$$\boxed{y = 4 - 16x^2 + \tfrac{16}{3}x^4.} \tag{2.6}$$

3 Bender and Orszag Problem 3.22

Given the following differential equation

$$x^3 y'' = y. \tag{3.1}$$

(a) Produce a solution in terms of a Taylor series at infinity of the form

$$y = \sum_n a_n x^{-n}. \tag{3.2}$$

Compute the coefficients a_n explicitly. There is only ONE linearly independent solution of this form.

(b) A second linearly independent solution of this equation has a leading order behavior $y \sim x$ when $x \to +\infty$. What is the next term in the expansion for x large and positive?

(c) Place your answers to (a) and (b) in terms of the classification of singular points.

Solution (a) A Taylor expansion at $x = \infty$ for the solution has the form

$$y = \sum_{n=0}^{\infty} a_n x^{-n}, \tag{3.3}$$

Substituting into the equation yields

$$(n+1)(n+2)\, a_{n+1} = a_n \quad \text{for} \quad n \geq 0, \tag{3.4}$$

with a_0 arbitrary. From this we obtain the convergent series for the solution given by:

$$\boxed{y = a_0 \sum_{n=0}^{\infty} \frac{x^{-n}}{n!\,(n+1)!}.} \tag{3.5}$$

This expansion has only one free parameter, hence there must be another solution which does not have a Taylor expansion series at ∞.

(b) From the problem statement we know that we can write the solution in the form

$$y = x + \varphi(x), \tag{3.6}$$

where $|\varphi(x)| \ll x$ when x is large. The equation for φ is thus

$$x^3 \varphi'' = x + \varphi. \tag{3.7}$$

In the limit of large x this can be approximated to

$$x^3\varphi'' \sim x. \tag{3.8}$$

This equation can be solved to give

$$\boxed{\varphi \sim -\ln x + c_0 + c_1 x,} \tag{3.9}$$

where c_0 and c_1 are constants. However, we **must** take $c_1 = 0$, since otherwise $|\varphi| \ll x$ for x large fails. We can also take $c_0 = 0$, since we can always subtract from this solution a multiple of the solution in (3.3 – 3.5), to set $c_0 = 0$.

<u>What is the next term?</u> To answer this question we write

$$y = x - \ln x + \psi(x) \tag{3.10}$$

with $|\psi| \ll |\ln x|$ for $x \gg 1$. Then

$$x^3\psi'' = -\ln x + \psi \sim -\ln x, \tag{3.11}$$

from which we obtain the leading order approximation

$$\psi \sim -\frac{\ln x}{2x}. \tag{3.12}$$

In solving for ψ, the same argument as above was used to set the constants to zero.

Continuing in this way, it is easy to see that an expansion for the solution involving terms of the form x^{-n} (for $n \geq -1$) and $(\ln x)x^{-m}$ (for $m \geq 0$) can be obtained.

(c) We use the transformation $x = 1/t$ to classify the point at ∞. This gives the differential equation

$$t\ddot{y} + 2\dot{y} = y, \tag{3.13}$$

from which it follows that $t = 0$, or equivalently $x = \infty$, is a *regular singular point*. Thus a solution should be provided by a Frobenius series of the form

$$y = t^s \sum_{n=0}^{\infty} a_n t^n. \tag{3.14}$$

Substituting this series into the differential equation (3.13) we get

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)\,a_n t^{n+s-1}+2\sum_{n=0}^{\infty}(s+n)\,a_n t^{n+s-1}=\sum_{n=1}^{\infty}a_{n-1}t^{n+s-1}. \tag{3.15}$$

Therefore we must have

$$s(s+1)\,a_0 = 0. \tag{3.16}$$

$$(n+s)(n+s+1)\,a_n = a_{n-1}, \qquad \text{for} \quad n \geq 1. \tag{3.17}$$

From this we can get only one solution (case $s=0$), which will be the same as the one obtained in part (a). In order to get a second solution we follow the procedure outlined in Bender and Orszag, and seek a solution with an expansion of the form

$$t^{-1}\sum_n t^n b_n - \ln t \sum_n t^n c_n = x\sum_n \frac{b_n}{x^n} + \ln x \sum_n \frac{c_n}{x^n}. \tag{3.18}$$

To sum up, ∞ is a regular singular point for the differential equation and therefore admits a Frobenius series expansion for at least one of the solutions. Series expansions can be found for the second solution as well, involving $\ln x$. The logarithm appears in the second solution, because the solutions to the indicial equation in (3.16) differ by an integer ($s=0$ and $s=-1$).

4 Bender and Orszag Problem 3.32

The differential equation $y'' + x^{-2}e^{1/x}\sin\left(e^{1/x}\right)y = 0$ **has an irregular singular point at** $x=0$. **Show that if we make the exponential substitution** $y=e^S$**, it is not correct to assume** $S'' << (S')^2$ **as** $x \to 0+$. **What is the leading behavior of** $y(x)$?

NOTE: This problem shows that dominant balance in the usual way does not always work. **EXPLAIN** why not.

However: a different dominant balance in the equation for S that results from the substitution $y=e^S$ gives an appropriate expansion for the solutions. **SHOW** this.

Solution Let $y = e^S$ and $\phi = S'$. Then

$$\phi' + \phi^2 + \underbrace{x^{-2} e^{1/x} \sin e^{1/x}}_{\left(\cos e^{1/x}\right)'} = 0. \tag{4.1}$$

As $x \to 0+$ the S-independent term oscillates extremely fast, with a very large amplitude. Assuming $\phi' << \phi^2$, ϕ can be found approximately by solving:

$$\phi^2 \sim -\left(\cos e^{1/x}\right)' \tag{4.2}$$

This implies that ϕ switches back and forth from real to imaginary very fast, with a large amplitude — completely nonsense, not even consistent with the assumption that $\phi' \ll \phi^2$.

If, on the other hand, we assume that $\phi^2 \ll \phi'$, then we get a consistent expansion

$$\phi = \phi_0 + \phi_1 + \phi_2 + ... \tag{4.3}$$

where $\phi_n = \mathcal{O}\left(x^n\right)$ as $x \to 0+$. In fact

$$\boxed{\phi_0 = c - \cos e^{1/x},} \tag{4.4}$$

where c is a constant. The next few terms satisfy the differential equations

$$\begin{aligned} \phi_1' &= -\phi_0^2, && (4.5) \\ \phi_2' &= -2\phi_0\phi_1, && (4.6) \\ \phi_3' &= -\left(\phi_1^2 + 2\phi_0\phi_2\right), && (4.7) \\ &\vdots \end{aligned}$$

In general, ϕ_n is given by

$$\boxed{\phi_n = -\int_0^x \left(\sum_{j=0}^{n-1} \phi_j \phi_{n-j-1}\right) dx.} \tag{4.8}$$

Using (4.8), we can prove by induction that $\phi_n = \mathcal{O}\left(x^n\right)$ as $x \to 0+$. This gives one family of solutions for ϕ, hence two for S upon integration.

5 Bender and Orszag Problem 3.1

We have used the transformation $x = 1/t$ to classify $x = \infty$ as an ordinary, regular singular or irregular singular point. Would the transformations $x = t^{-n}$, $x = t^{-\alpha}$ $(\alpha > 0)$, $x = 1/\sinh t$ also work? Is there any advantage to $x = 1/t$?

Solution For the purpose of classifying the point at ∞ as ordinary, regular singular, etc., one needs a transformation $x \to t$ — **in the complex plane** — that

(a) sends $x = \infty$ to some finite point, say $t = 0$,

(b) is analytic in a neighborhood of $x = \infty$, and

(c) is single valued and invertible near $x = \infty$.

From (a-c) the transformation will have a Laurent expansion of the form

$$t = c_0\, x^{-1} + c_1\, x^{-2} + c_2\, x^{-3} + \dots \tag{5.1}$$

with $c_0 \neq 0$, which converges for $|x| > R$, for some $R \geq 0$. Alternatively, the inverse function will have a series of the form

$$x = c_0\, t^{-1} + a_1 + a_2\, t + \dots \tag{5.2}$$

converging for $0 < |t| < r$, for some $r > 0$.

Any function with these properties will do (and will give the same answer), but the simplest one is $1/x$. The alternative given in the problem statement $x = t^{-\alpha}$, $\alpha > 0$ cannot satisfy all of (a), (b) and (c) above unless $\alpha = 1$. The transformation $x = 1/\sinh(t)$ will work, but choosing $x = 1/t$ is a lot simpler.

6 Bender and Orszag Problem 3.33 (a)

Find the leading behavior, as $x \to 0+$, for the following equation

$$x^4 y''' = y. \tag{6.1}$$

Solution Let $y = e^S$ and $\phi = S'$. Then (6.1) becomes

$$\phi'' + 3\phi\phi' + \phi^3 = x^{-4}. \tag{6.2}$$

Assume that ϕ^3 dominates on the left. Then

$$\phi \sim \sigma x^{-4/3} \tag{6.3}$$

where σ is a cubic root of 1 $(\sigma^3 = 1)$. This yields 3 solutions as needed. We now check the solution for consistency

$$\begin{aligned} \phi'' &= \mathcal{O}\left(x^{-10/3}\right) \ll \phi^3 = \mathcal{O}\left(x^{-4}\right). && (6.4)\\ \phi\phi' &= \mathcal{O}\left(x^{-11/3}\right) \ll \phi^3 = \mathcal{O}\left(x^{-4}\right). && (6.5) \end{aligned}$$

The complete expansion for ϕ is then given in powers of $x^{-1/3}$ as follows

$$\phi \sim \sigma x^{-4/3} + a x^{-1} + b x^{-2/3} + \ldots \tag{6.6}$$

By substituting the form above and collecting equal powers of x we find

$$1 = x^4\left(\phi'' + 3\phi\phi' + \phi^3\right) = \sigma^3 + \sigma^2(3a-4)\,x^{1/3} + \sigma\left(3b\sigma + 3a^2 + \frac{28}{9} - 7a\right)x^{2/3} + \ldots \tag{6.7}$$

From this we conclude that

$$\begin{aligned} a &= 4/3, && (6.8)\\ b &= 8\sigma^{-1}/27, && (6.9) \end{aligned}$$

and so on. Therefore, for some constant c_0,

$$S \sim \underbrace{-3\sigma x^{-1/3} + \frac{4}{3}\ln x + c_0}_{\text{dominant terms}} + \underbrace{\frac{8}{9\sigma}\,x^{1/3} + \ldots}_{\text{small}} \tag{6.10}$$

It follows that the leading behavior of y as $x \to 0+$ is given by

$$\boxed{y \sim c\, x^{4/3}\, e^{-3\sigma x^{-1/3}},} \tag{6.11}$$

where c is some constant.

<u>Note:</u> To obtain the form for ϕ in equation (6.6), we proceed in the usual way. First write

$$\phi = \sigma\, x^{-4/3} + w_1\,(x)\,, \tag{6.12}$$

where $w_1 = o(x^{-4/3})$ as $x \to 0+$. Then substitute this into equation (6.2), to obtain an equation for w_1. A dominant balance analysis then shows that $w_1 \sim 4/(3\,x)$. Hence write

$$\phi = \sigma\, x^{-4/3} + \frac{4}{3\,x} + w_2\,(x)\,, \tag{6.13}$$

where $w_2 = o(x^{-1})$ as $x \to 0+$. Then we substitute this into the equation for ϕ, to obtain an equation for w_2, and (again) do a dominant balance analysis. And so on and so forth.

Of course, after you have some experience with doing expansions (meaning: you have done gazillions of them), you will be able to guess the form in equation (6.6) — hence short-cutting the painful process sketched above.

THE END.