This is an integral of a complex exponential:

\[ I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x - x^5/5} \, dx, \quad \lambda \gg 1 \]

The method of Stationary Phase may be used to calculate the leading term of this integral. The points of stationary phase is where \( v(x) \) doesn't change:

\[ v'(x) = 1 - x^4 = 0 \]

which is valid for \( x_0 = \pm 1, \pm i \). But only \( x_0 = \pm 1 \) matters, since we are integrating along the real axis:

\[ v''(x) = -4x^3 \]

Expanding \( v(x) \) around \( x_0 = \pm 1 \) (Remember \( v'(x_0) = 0 \)):

\[ v(x) \approx v(x_0) + \frac{1}{2} v''(x_0) (x - x_0)^2 + O(dx^3) \]

Plug this Taylor expansion and evaluate the integral on both \( x_0 \):
At $x_0 = \frac{4}{5}$, the contribution is:

$$\nu'(1) = \frac{4}{5}, \quad \nu''(1) = -4 \quad \text{(negative $\nu''(x_0)$)}$$

$$e^{-i\pi/4} \sqrt{\frac{2\pi}{\lambda \nu''(x_0)}} e^{i\lambda \nu(x_0)} h(x_0) = e^{-i\pi/4} \sqrt{\frac{\pi}{2\lambda}} - e^{i\lambda \frac{4}{5}}$$

At $x_0 = -1$, the contribution is:

$$\nu(-1) = \frac{1}{5}, \quad 1 = -\frac{4}{5} \quad \nu''(-1) = 4 \quad \text{(positive $\nu''(x_0)$)}$$

$$e^{i\pi/4} \sqrt{\frac{2\pi}{\lambda \nu''(x_0)}} e^{i\lambda \nu(x_0)} h(x_0) = e^{i\pi/4} \sqrt{\frac{\pi}{2\lambda}} - e^{-i\lambda \frac{4}{5}}$$

The integral is, finally:

$$I(\lambda) \approx \sqrt{\frac{\pi}{2\lambda}} \left[ e^{i\left(\frac{4\lambda}{5} - \frac{\pi}{4}\right)} + e^{i\left(\frac{4\lambda}{5} + \frac{\pi}{4}\right)} \right]$$

$$I(\lambda) \approx \sqrt{\frac{\pi}{2\lambda}} \cdot 2 \cdot \cos \left(\frac{4\lambda}{5} - \frac{\pi}{4}\right)$$

$$I(\lambda) \approx \sqrt{\frac{2\pi}{\lambda}} \cdot \cos \left(\frac{4\lambda}{5} - \frac{\pi}{4}\right)$$
1) \[ I(\lambda) = \int_{-\infty}^{\infty} e^{\lambda x} \cdot e^{-x^4/4} \, dx, \quad \lambda > 1 \]

To be honest, I was reading the book and this is solved in Example 6.71, so here goes.

We have \[ I(\lambda) = \int_{-\infty}^{\infty} e^{\lambda w(x)} \, w(x) \, dx, \quad \text{where} \quad w(x) = x, \quad h(x) = e^{-x^4/4} \]

Since \( w'(x) = 1 \neq 0 \text{ always and there are no finite end points, we expect the main contribution to come from a saddle point in the complex plane.} \]

Let's normalize the integrand:

\[ i(\lambda x^4 - x^4) = i\lambda \left( x + \frac{i x}{4\lambda} \right) \]

Now we say \( \lambda \cdot t + i \lambda \cdot \frac{t^4}{4} \), so we need \( 4c - 1 = c \)

\[ c = 1/3 \]

Alright, so we call \( x = \lambda^{1/3} t \) and \( dx = \lambda^{1/3} dt \), the integral becomes:

\[ I(\lambda) = \lambda^{1/3} \int_{-\infty}^{\infty} e^{\lambda(t + i t^4/4)} \, dt, \quad \text{where} \quad \Lambda = \lambda^{4/3} \]
To find the saddle points we must calculate and the derivatives \( f(t) = t + \frac{i t^4}{4} \) go to zero:

\[
f'(t) = 1 + i t^3, \quad f'(t_0) = 1 + i t_0^3 = 0
\]

The roots are:

\[
t_0 = -i, \quad t_0 = e^{i\pi/6} \approx 0.87 + 0.5i \quad t_0 = e^{i3\pi/6} = -0.87 + 0.5i
\]

To find out which saddle points matter we must do:

- \( t_0 = -i \):
  \[
f(t_0) = -i + i \left( (-i)^4 \right) = -3i
  \]
  \[
  I(\lambda) = \lambda^3 \int_{-\infty}^{\infty} e^{\frac{-i\lambda 3i}{4}} dt
  = \lambda^3 \int_{-\infty}^{\infty} e^{\frac{-\lambda^3}{4}} dt
  
  \]
  
  The integral is exponentially large, so
  \( t_0 = -i \) isn't a relevant saddle point.
\[ t_0 = e^{\frac{i\pi}{6}} \quad , \quad f(t_0) = \frac{3}{4} e^{-\frac{i\pi}{6}} \rightarrow \text{This point isn't exponentially large for the integrand.} \]

\[ t_0 = e^{i\frac{5\pi}{6}} \quad , \quad f(t_0) = -\frac{3}{4} e^{-i\frac{\pi}{6}} \rightarrow \text{This point is also relevant.} \]

Following the example's advice, instead of trying to draw the level curves of \( u \) and \( v \) \((t = u + iv)\), we'll just deform the whole curve to \( t = u + \frac{1}{2}i \):

To justify this choice we calculate \( f(t) \) on this line:

\[
 f(T + \frac{i}{2}) = u(T) + i v(T) = T + \frac{i}{2} + \frac{i}{4} \left( T^4 + 4 \cdot \frac{T^3}{2} + 6 T^2 \frac{i}{4} + 4 T \frac{i^3}{8} + \frac{i^4}{16} \right) =
\]

\[
 = T + \frac{i}{2} + \frac{i T^4}{4} - \frac{T^3}{2} - \frac{3 T^2 i}{8} + \frac{T}{8} + \frac{i}{64} =
\]

\[
 = \left( \frac{9T}{8} - \frac{T^3}{2} \right) + i \left( \frac{T^4}{4} - \frac{3 T^2}{8} + \frac{33}{64} \right)
\]

\[ u(T) \quad \quad \quad \quad v(T) \]
\[ n''(T) = 3T^2 - \frac{3}{4}, \text{ which is positive at the two saddle points, also } n'(T) = T^3 - \frac{3}{4}T = 0 \text{ at saddle points. This means the integrand is minimum on both saddle points.} \]

Doing the Taylor expansion near both saddle points:

\[ f(t) = (e^{i\pi/6} + 4i e^{i\pi/3}) + f'(e^{i\pi/6})(t - e^{i\pi/6}) + \frac{f''(t_0)}{2}(t - e^{i\pi/6})^2 \]

\[ = \left[ \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{i}{4} \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \right] + \left( 1 + i e^{i\pi/2} \right)(t - e^{i\pi/6}) + \frac{3}{2} e^{-i\pi/6} (t - e^{i\pi/6})^2 \]

\[ i \wedge f(t) = \frac{3}{8} \wedge (i\sqrt{3} - 1) - \frac{3}{2} e^{i\pi/3} (t - e^{i\pi/6})^2 \]

The integral becomes for \( t_0 = e^{i\pi/6} \):

\[ I(\lambda) = \lambda^{i\pi/6} e^{\frac{3}{8} \wedge (i\sqrt{3} - 1)} \int_{-\infty}^{\infty} e^{\frac{3}{2} e^{i\pi/3} (t - e^{i\pi/6})^2} \ dt \]

We may solve this Gaussian integral with a change of variables: \( y = t - e^{i\pi/6}, \ dy = dt \)
\[ \int_{-\infty}^{\infty} e^{-\frac{3}{2} \lambda} e^{\frac{i\pi}{3} y^2} dy = \sqrt{\frac{\pi}{\frac{3}{2} \lambda e^{i\pi/3}}} \]

Finally, the contribution from the to-e^{i\pi/6} saddle point:

\[ \lambda^{1/3} \sqrt{\frac{2\pi}{\frac{3}{8} \lambda}} e^{i\frac{\pi}{3}} \left( \frac{3\lambda^{1/3} - \pi}{2} \right) \]

Considering that the other saddle point e^{i3\pi/6} will just yield the complex conjugate of the above, then:

\[ I(\lambda) \approx 2 \lambda^{1/3} \sqrt{\frac{2\pi}{\frac{3}{8} \lambda}} e^{i\frac{\pi}{3}} \left( \frac{3\lambda^{1/3} - \pi}{2} \right) \]

with \( \lambda = \lambda^{1/3} \) we get:

\[ I(\lambda) \approx \sqrt{\frac{8\pi}{3}} \sqrt{\lambda} e^{-\frac{3}{8} \lambda^{1/3}} \cos \left( \frac{3\lambda^{1/3}}{8} \lambda^{1/3} - \frac{\pi}{6} \right) \]

Too long.
2) \( I(\lambda) = \int_{0}^{\infty} e^{-\lambda x} e^{-\frac{1}{\lambda x^3}} \, dx \), \( \lambda > 1 \)

To evaluate this real integral we may use Laplace's method, but let's first normalize it:

\[-\lambda \left( x + \frac{1}{\lambda x^3} \right), \text{ say } x = \lambda^c t, \text{ then }\]

\[
\left( x + \frac{1}{\lambda x^3} \right) = \lambda^c t + \frac{\lambda^{3c-1}}{\lambda} \cdot t^3, \text{ we need } c = -3c - 1, \quad c = -\frac{1}{4}
\]

So, \( x = \lambda^{\frac{1}{4}} t \), \( dx = \lambda^{\frac{1}{4}} dt \), limits remain \((0, \infty)\)

\[
I(\lambda) = \lambda^{\frac{1}{4}} \int_{0}^{\infty} e^{-\lambda^{\frac{3}{4}}(t + \frac{1}{t^3})} \, dt
\]

The greatest contribution is when the integrand is minimum:

\[
f(t) = t + t^{-3}
\]

\[
f'(t) = 1 - 3t^{-4}, \quad f'(t_0) = 1 - 3t_0^{-4} = 0, \quad t_0 = 3^{\frac{1}{4}}
\]

\[
f''(t) = 12t^{-5}, \quad f''(t_0) = \frac{12}{3^{5/4}} > 0 \quad \text{so } t_0 \text{ is indeed a minimum}
\]
Taylor series expansion around \( t_0 = 3^{1/4} \):

\[
f(t) = f(t_0) + \frac{f'(t_0)}{1!}(t-t_0) + \frac{f''(t_0)}{2!}(t-t_0)^2 + \cdots
\]

\[
= f(t_0) - \frac{3^{1/4}}{2!} + \frac{f''(t_0)}{2!}(t-t_0)^2 + \cdots
\]

Thus, the integral:

\[
I(x) = \int e^{-t^2} \, dt
\]

If we call \( y = t - 3^{1/4} \), \( dy = dt \), then

\[
I(x) = \int e^{-y^2} \, dy
\]

Since this integrand quickly decays away from \( t_0 \), we may set the limits to \( -\infty \) without much error.

Solving this Gaussian integral yields:

\[
I(x) \approx \left( \frac{\sqrt{\pi}}{2^{3/4}} \right) e^{-x^2/2} \int_0^\infty e^{-y^2/2} \, dy
\]

\[
= 2^{3/4} \frac{\sqrt{\pi}}{2} \sqrt{\frac{3}{\pi}}
\]

\[
= e^{-A(H/3)^{3/4}} \left[ \frac{\pi}{2} \sqrt{A(\lambda)} \right]^{3/4}
\]

\[
= e^{-A(\lambda^{3/4})^{3/4}}
\]

\[
= -A e^{-A(\lambda^{3/4})^{3/4}}
\]
Let's split the integrand into the two cases 

\[
\phi \left( \phi \sum_{2\pi} \frac{e}{\gamma \sin \gamma} \right) = \int_{-\pi/2}^{\pi/2} \phi \sum_{2\pi} \frac{e}{\gamma \sin \gamma} \, d\gamma
\]

and we need to evaluate the integral using asymptotics.

And we are interested in finding \( x \), for which \( Z(x) = 0 \).

3) The Bessel function:

\[
\phi \left( \phi \sum_{2\pi} \frac{e}{\gamma \sin \gamma} \right) = \int_{-\pi/2}^{\pi/2} \frac{1}{(x)} \, d\gamma
\]
Binomial expansion correction:

\[(\cos \phi)^{2n} = \frac{1}{4^n} (e^{i\phi} + e^{-i\phi})^{2n} = \frac{-i2^{2n}\phi}{4^n} (1 + e^{i2\phi})^{2n}\]

Remember how \((1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k\), so:

\[(1 + e^{i2\phi})^{2n} = \sum_{k=0}^{2n} e^{ik\phi} \binom{2n}{k}\] and \((\cos \phi)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} e^{ik\phi}\)

So our integral becomes:

\[I(\lambda) = \sum_{k=0}^{2n} \frac{1}{4^{n} \binom{2n}{k}} \int_{-\pi/2}^{\pi/2} e^{i(\lambda \sin \phi + z \phi(k-\phi))} d\phi\]

We have already solved this integral using saddle point:

\[I(\lambda) = \sum_{k=0}^{2n} \frac{2}{4^{n} \binom{2n}{k}} \frac{2\pi}{\lambda} \cdot \cos \left( x + \pi(k-\phi) - \frac{\pi}{4} \right)\]

where \(k < 2n\)

Finally, the large zeros happen when the cosines are zero:

\[\cos \left( x_0 + \pi(k-\phi) - \frac{\pi}{4} \right) = 0\]

\[x_0 + \pi(k-\phi) - \frac{\pi}{4} = m\pi\]

\[x_0 = \pi \left( \frac{1}{4} + \phi + m - k \right)\]

\[m < k\]

\[k < 2n\]