The Lovász Local Lemma and Applications

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Abstract

A true gem in the probabilistic method is the Lovász Local Lemma, which for a set of bad events $A_1, \ldots, A_m$ with bounded dependence degree can guarantee that the event $\bar{A}_1 \wedge \ldots \wedge \bar{A}_m$ has positive probability. In this paper we give a proof of the generalized form and show surprising applications of this powerful method to Ramsey theory, hypergraph coloring and the satisfiability problem.

1 Introduction

One of the fundamental techniques in combinatorics is the probabilistic method. To show the existence of a desired object, for example an n-node graph without any $4\log n$-size clique or anti-clique, one considers a suitable random experiment to show it's existence. . . .

The first application of this technique can be found in . . ., where it was used to show. . . . The method was also used in the groundbreaking work of Leighton, Maggs and Rao [2] in the context of packet routing. For a more detailed overview, we refer to the excellent book of Alon and Spencer [1].

The local lemma is essentially an existential result and does not give an efficient algorithm by itself. However, recently, Moser proved a general algorithmic local lemma that can be applied to numerous situations to give a randomized algorithm of finding an instance that satisfies $\bar{A}_1 \wedge \ldots \wedge \bar{A}_m$.

In this paper, I will first state the local lemma and present its proof. Then I will review several applications of the local lemma. Finally, I will conclude with the proof of Moser's result.

2 Local Lemma

State local lemma and present its proof

Theorem 2.1. *Let $A_1, A_2, \cdots, A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = [n]$ is called a dependency digraph for the events*

$A_1, \ldots, A_n$ if for each $i \in [n]$, the event A_i is mutually independent of all the events $\{A_j : (i,j) \notin E\}$. Suppose that $D = (V, E)$ is a dependency digraph for the above events and suppose that there are real numbers $x_1, \ldots, x_n$ such that $0 \leq x_i < 1$ and

$$\mathbf{P}(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j) \forall i \in [n].$$

Then with $\mathbf{P}(\bigwedge_{i=1}^{n} \overline{A_i}) \geq \prod_{i=1}^{n}(1 - x_i)$. In particular, with positive probability no event A_i holds.

3 Applications of local lemma

3.1 Application I

First application is the original application of Erdős and Lovász. I will study k-coloring of natural numbers.

Theorem 3.1. *State theorem*

3.2 Application II

Second application is the work of Leighton, Maggs and Rao [2].

Theorem 3.2. *State theorem*

3.3 Application III

Third application is property B of hypergraphs.

Theorem 3.3. *State theorem*

4 Algorithmic local lemma

I will study the result of Moser; algorithmic version of the local lemma.

Theorem 4.1. *State theorem*

References

[1] Noga Alon and Joel H. Spencer, *The probabilistic method*, 3rd ed., Wiley, New York (2008).

[2] Frank T. Leighton, Bruce M. Maggs, Satish B. Rao, Packet routing and job-shop scheduling in O(congestion+ dilation) steps, *Combinatorica* **14** (1994), 167–186.