6. The regular Bessel function $J_n(x)$ sometimes appears through an integral representation

$$J_n(x) = \frac{1}{\pi i^n} \int_0^\pi d\theta e^{ix\cos(\theta)} \cos n\theta.$$  

(a) Use differentiation under the integral and other appropriate steps to show that the function defined by the integral satisfies the Bessel ODE.

(b) Argue that the first $n$ power series coefficients of the function defined by the integral vanish, so that $J_n(x) \sim x^n$ as $x \to 0$.

7. Consider the Bessel equation of order 1/2 on the interval $[1, 2]$ with Dirichlet boundary conditions

$$x^2 y'' + xy' - \frac{1}{4} y = -k^2 x^2 y, \quad y(1) = y(2) = 0.$$  

(a) Show that $y_1 = \sin kx / \sqrt{x}$ and $y_2 = \cos kx / \sqrt{x}$ are two independent solutions of the equation.

(b) Find the eigenvalues and the corresponding eigenfunctions.

(c) Change the boundary conditions to $y(1) = y'(2) = 0$ and show that the eigenvalues are the roots of the equation $\tan k = 4k$.

8. In class we proved results for the reality and sign of eigenvalues in Sturm-Liouville problems for pure boundary conditions. Here you are asked to extend them, if possible to the case of mixed boundary conditions $\phi(a) + k_a \phi'(a) = 0$ and $\phi(b) + k_b \phi'(b) = 0$.

(a) Show that eigenvalues $\lambda$ must be real for all $k_a$, $k_b$.

(b) Find conditions on $k_a$, $k_b$ which are sufficient to prove $\lambda > 0$. You may assume that $q(x) \leq 0$.

9. Generalize the Sturm comparison theorem to study problems with general boundary condition $\phi(a) + k_a \phi'(a) = 0$. Show that the eigenfunction $\phi_{n+1}(x)$ has at least one zero between $x = a$ and the first zero of $\phi_n(x)$.

Problem 10 on reverse page.
10. The Gram-Schmidt orthogonalization procedure is a systematic method to start with a (finite) set of linearly independent vectors \( v_1, v_2, \ldots, v_n \) and produce a set of orthogonal vectors \( e_1, e_2, \ldots, e_n \). The orthogonal set has the property \( e_k = \sum_{i=1}^{k} a_{i,k} v_i \) with definite coefficients \( a_{i,k} \). Two convenient on-line references to the procedure are: ** en.wikipedia.org/wiki/GramSchmidt_process, ** and ** www.math.hmc.edu/calculus/tutorials/gramschmidt/gramschmidt.pdf. **

In this problem, students are asked to apply this procedure in two situations, the first for the vector space \( \mathbb{R}^3 \) and the second for the function space \( C[-1, 1] \).

(a) Start with the three independent vectors \( v_1 = (1, 0, 0), \ v_2 = (1, 2, 0), \ v_3 = (1, 2, 3) \) and produce an orthogonal set \( e_1, e_2, e_3 \) with the properties above.

(b) In the function space \( C[-1, 1] \), start with the four monomials \( v_0 = 1, \ v_1 = x, \ v_2 = x^2, \ v_3 = x^3 \) and produce a set of \( n \)th order polynomials \( P_n(x), n = 0, 1, 2, 3 \) which satisfy \( P_n(1) = 1 \) and the orthogonality condition \( \int_{-1}^{1} dx P_m(x) P_n(x) = 2 \delta_{mn}/(2n + 1) \). Check that your answers agree with the standard Legendre polynomials, see page 340 of Haberman. If not fix your mistakes with the Gram-Schmidt procedure. Then show explicitly that \( P_2(x) \) and \( P_3(x) \) satisfy the Legendre differential equation (for the appropriate value of \( n \)).