\[ G(x) = -\frac{1}{(2\pi)^3} \frac{2}{|x|} \int_0^\infty \frac{dk}{k} \sin k |x| \]

Put things together

\[ G(x) = -\frac{1}{2\pi |x|} \int_0^\infty \frac{dv}{v} \sin v v = k|x| \]

The remaining integral is simply a constant, but let's evaluate because it is a nice example of contour methods.

\[ H = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dv}{v} \sin v \quad \text{true since } \cos v = \frac{\sin v}{v} \quad \text{and even} \]

Thus \( f(v) \) has no poles anywhere.

("Pole at } v=0 \text{ cancels}")

Cauchy's Theorem allows to distort the contour.

\[ H = \frac{1}{2} \int_C \frac{dv}{v} \]

The rigorous reason that \( \int_{-\infty}^{\infty} \frac{dv}{v} = \int_{C_+} \frac{dv}{v} \]

\[ H = \frac{1}{4i} \int_C \left[ \frac{e^{iv}}{v} - \frac{e^{-iv}}{v} \right] dv \]

\[ = H_+ + H_- \]

Complete \( H_+ \)

Since there are no poles, we have \( \lim_{v \to \infty} \int_{C_+} e^{iv} dv = 0 \)

Complete \( H_- \)

Since \( \lim_{v \to \infty} \int_{C_-} e^{-iv} dv = 0 \)

\[ H_- = -\frac{1}{4i} \int_{C_-} e^{-iv} dv \]

\[ = -\frac{1}{4i} (2\pi i) = -\frac{\pi}{2} \]

Finally \( G(v) = G(r) = -\frac{1}{4\pi r} \)
Thus it totally expected from viewpoint of physics. It says that
the electric potential of a unit point charge at the origin \( \mathbf{r} = 1/r \).

**Example**: Gradient of any \( f(r) \) is \( \nabla f(r) = \mathbf{\hat{r}} f'(r) \)

Electric Field \( \mathbf{E} = -\nabla V(x) = -\frac{1}{2\pi} \frac{Q}{r^2} \)

Inverse square law force.

We can use \( \nabla \cdot \mathbf{E} = 0 \) to write set of Poisson's

\[ \nabla^2 U(x) = 0 \]

for any source \( \rho \),

\[ U(x) = \frac{1}{4\pi} \int \frac{\rho(y)}{|x-y|} \, dy \]

(i) integral converges as \( y \rightarrow 0 \) since \( \frac{1}{|x-y|} \) is an integrable

sing. for \( D=3 \). In coord. centered at \( x \)

\[ \int \frac{d^3y}{|x-y|} = \int \frac{r^2 dr}{r} \]

(ii) it converges as \( y \rightarrow \infty \) if \( p(y) \) vanish suff. fast.

We want a situation when total charge on mass \( m \) finite

\[ \int d^3y \rho(y) = \int r^2 dr d(\cos \theta) d\phi \rho \phi(y) < \infty \]

Converges if \( |\rho(y)| \leq \frac{C}{|y|^{3+\varepsilon}} \)

Then \( \int \frac{d^3y}{|x-y|} \) converges even faster.

**Emphasis**: \( G = -\frac{1}{4\pi} \frac{1}{|x-y|} \) is both translation invariant,

because \( R^3 \) is both homogeneous and isotropic.

A simpler way to say this: the response at \( x \) to a pt charge at \( y \)

depends only distance \( |x-y| \).

**Study an example of Poisson's eq** (stcl. in physics courses)

\[ \nabla^2 U(x) = \begin{cases} \rho(r) & r < R \\ 0 & r > R \end{cases} \] bounded sph. sym.

b.sph. sym.

Source


potential due to mass. distrib. inside

the earth.
Write 
\[ \hat{x} = r \hat{r} \]
\[ \hat{y} = r \hat{\theta} \]
\[ \hat{z} = \hat{\phi} \]
\[ \hat{x}^2 + \hat{y}^2 = 1 \]

There are techniques to do this triple integral, but it is more intelligent to solve PDEs directly using
\(c)\) sph. sym. of problem

**Green's Theorem**
\[ \int_V d^3 x \, \nabla \cdot \mathbf{F} = \int_S d^2 x \, \hat{n} \cdot \mathbf{F} \]

Sph sym \(\Rightarrow\) \[ U(\mathbf{r}) = \Phi(r) \]

Let \( \mathbf{F} = \nabla u \) so PDE is

\[ \nabla \cdot \mathbf{F} = \begin{cases} \rho(r) & r < R \\ 0 & r > R \end{cases} \]

For any \( V \) bounded by surface \( S \)
\[ \int_S d^2 x \, \hat{n} \cdot \nabla u = \int_V d^3 x \, \nabla \cdot \mathbf{F} \]

Most useful choice is to take \( V \) to be a ball with center at origin, radius \( r_0 \)
\[ \int = \int_{r_0}^{r_0} r^2 dr \rho(r) \Theta(r-r) \int_0^{2\pi} \int_0^\pi \sin \theta d \theta d \phi \]
\[ = 4\pi \]

Mass contained in sph. of radius \( r_0 \)
\[ M(r_0) \] of \( r_0 \geq R \), then \( M(r_0) = M = \) total mass

Now we need to evaluate surface integral. From Ex. 6

a) \( \hat{n} \cdot \nabla u = \hat{n} \cdot \hat{r} = u'(r) = \Phi'(r) \)
b) use sph. coords
\[ d^2 x = r_0^2 \sin \theta d \theta d \phi \]
c) \[ \int_S d^2 x \, \hat{n} \cdot \nabla u = r_0^2 u'(r_0) = M(r_0) \]
Thus \( u'(r_0) = \frac{M(r_0)}{4\pi r_0^2} \), since no particle, we will let \( r_0 \to r \) and write result \( u'(r) = \frac{M(r)}{4\pi r^2} \).

This means that gravitational force \( \vec{F} = -\nabla u = -\hat{x} u'(r) \) is

\[
\vec{F} = -\frac{1}{4\pi} \frac{M(r)}{r^2} \hat{x}
\]

(c) If \( r > R \), \( M(r) = M_{\text{Earth}} \). Outside the earth, we find exact \( \frac{1}{r} \) attractive inverse square law force of Newtonian gravity.

(c) If \( r < R \), the attraction decreases as we move inward.

The force one feels at any \( r \) is attractive \( \frac{1}{r^2} \) force due to the amount of mass between \( dM \) and origin.

(The net force due to mass at \( r_0 < r < R \) can also be found)

Another receipt on Green's for \( \nabla^2 u \) in \( \mathbb{R}^d \). We will derive result by a very simple method which works for all \( d \). Illustrate power of symmetry and dimension analysis.

Proof: \( \nabla^2 G^{(d)}(\mathbf{x} - \mathbf{y}) = \delta^{(d)}(\mathbf{x} - \mathbf{y}) \) in \( \mathbb{R}^d \).

(a) Use trace of \( \mathbf{y} = 0 \), \( \Rightarrow \nabla^2 \delta^{(d)}(\mathbf{y}) = \delta^{(d)}(\mathbf{y}) \).

(b) there are no dimensional fixed parameters in the solution. (in fact no parameters, at all). We must construct solution in terms of coads \( x_1, x_2, \ldots, x_d \).

(c) rotational symmetry \( \Rightarrow G \) can only depend on \( \mathbf{x} \) distance from origin \( r = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2} \).

(d) Some dimensional analysis \([\nabla^2] = \frac{1}{r^2}\)

\[
[\delta^{(d)}(\mathbf{x})] = \frac{1}{r^d} \quad \text{since} \quad \int d^d \kappa \delta^{(d)}(\mathbf{y}) = 1
\]

From PDE \( \nabla^2 G = \delta \), we find \([G^{(d)}] = \frac{1}{r^{d-2}}\)

(e) So the Green's function \( G(d) \) is \( \frac{1}{r^{d-2}} \) for \( d = 2 \).

For \( d > 2 \), the only possibility is \( \frac{1}{r^{d-2}} \)
so sym. and dim. study, has determined $C_d$ for up to a const factor $C_d$.

What about $d=2$, dim and sym allows $G^{(2)} = C$, but then $\nabla \cdot G^{(2)} = 0$ not $G^{(2)}(\mathbf{x})$.

We need something else. Look at $\nabla \cdot G^{(2)}(\mathbf{x})$, rather than $G^{(2)}$.

Apply sym + div, c.f. (3.5) to $\nabla \cdot G^{(2)}$:

$$\frac{1}{q} \frac{\partial}{\partial q} = \frac{1}{q^2} \Rightarrow \left[ \nabla G^{(2)} \right] = \frac{1}{q}$$

Rotation sym $\Rightarrow G^{(2)} = f(r)$,

then $\nabla G^{(2)} = \frac{df}{dr} \nabla r = f'(r) \mathbf{r}$ as in Ex. 3.1.

Since $[f'] = \frac{1}{r}$, we have src. $f'(r) = C_2/r$.

Then $f(r) = C_2 \ln (r/r_0)$. Think of $C_2$ as an "integ. const" necessary to make argument of $\ln(\cdot)$ dimensionless. We will discuss this soon.

How do we find the constants $C_d$? Apply Gauss law to a ball of radius $R$, about origin:

$$\int_{V_d} \int dV \cdot \nabla \cdot \mathbf{E} = \int_{S_{d-1}} dS \mathbf{E} \cdot \mathbf{n} = \int_{S_{d-1}} dS = \int_{S_{d-1}} dS$$

LHS $= \int_{V_d} \int dV \cdot \nabla^2 G = \int_{V_d} dV \cdot \oint_{S_{d-1}} dS = 1$

RHS $\mathbf{n} \cdot \nabla G = G'(r) = C_d \frac{3}{r} \frac{1}{d-1} \left\{ \begin{array}{c}
\frac{1}{r^{d-2}} & d > 2 \\
\frac{1}{r} & d = 2 \\
-\frac{(d-2)}{r^{d-1}} & d > 2
\end{array} \right.$

So LHS $= C_d \frac{3}{R^{d-1}} \left\{ \begin{array}{c}
\int_{S_{d-1}} dS & d = 2 \\
\left\{ \begin{array}{c}
\frac{1}{2} \pi r^2 & d = 2 \\
1 & 2 \\
1 & d > 2
\end{array} \right. 
\end{array} \right.$

Def: $\Omega_d = \frac{1}{R^{d-1}} \int_{S_{d-1}} dS$ = surf. area of unit sph in $d$ dim:

For $d=2$ we know $\Omega_2 = 2\pi$

$d=3 \quad \Rightarrow \quad \Omega_3 = 4\pi$
For general $d$: $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ can be obtained from integral of $e^{-\frac{r}{\xi}}$ in $d-$dim.

Conclusion from Gauss Law:

$$d = 2, \frac{1}{\Omega_d} = 4\pi \int_{d=2}^{d>2}$$

Our final result for Green's fn:

$$d > 2 \quad G^{(d)}(x) = -\frac{1}{(d-2)\Omega_d} \frac{1}{r^{d-2}}$$

$$d = 2 \quad G^{(2)}(x) = \frac{1}{2\pi} \ln\left(\frac{|x|}{\xi_0}\right)$$

Using $G^{(d)}$ we can write the soln of $\nabla^2 u = \delta(x)$ in $u(x) = \int d^d y G^{(d)}(x-y) \rho(y)$

In $d=2$ $u(x) = \frac{1}{2\pi} \int d^2 y \rho(y) \ln\left(\frac{|x-y|}{\xi_0}\right)$

What is effect of $\xi_0 $ which is quite artificial + arbitrary. The $\xi_0$ term just gives $-\ln(\xi_0) \int d^2 y \rho(y)$. It adds a constant to potential $u(x)$. In physical situations, an additive constant in potential has no physical effect. Rather it is the force $\vec{F}$ which is physical and $\nabla u$ does not depend on $\xi_0$.

A practical + interesting app. of $G^{(d)}$: u to Image charge method for $1/2$ space probs.

I. Find $G_{d,11}(x_1, x_2; y_1, y_2)$ on upper half plane $x_0 > 0$

$$\nabla^2 G_{d,11}(x_1, x_2; y_1, y_2) = \delta(x_1-y_1) \delta(x_2-y_2)$$

$$G^{(d)}(x_1, 0, y_1, y_2) = 0$$

$$\frac{\partial}{\partial y_2} G(1, 0, y_1, y_2) = 0$$

Reason as follows: a) full space $d^2 = \frac{1}{4\pi} \ln\left(\frac{|x_1-y_1|^2 + (x_2-y_2)^2}{\xi_0^2}\right)$ satisfies PoE with correct pt. source at $(y_1, y_2)$ but not BC.
b) Add an image charge at \((y_1, y_2)\) to find complete sol. +

\[
G_0(x_1, x_2, y_1, y_2) = \frac{1}{4\pi} \left( \frac{\ln \left( \frac{(x_1-y_1)^2 + (x_2-y_2)^2}{r_{02}} \right)}{(y_1-y_2)^2 + (x_2+y_2)^2} \right)
\]

\[
= \frac{1}{4\pi} \ln \left( \frac{(x_1-y_1)^2 + (x_2-y_2)^2}{(y_1-y_2)^2 + (x_2+y_2)^2} \right) r_0 \text{ drops out.}
\]

\[
\text{as } \gamma_y \to 0, \quad G_0 \to \frac{1}{4\pi} \ln \left( \frac{(x_1-y_1)^2 + y_2^2}{(y_1-y_2)^2 + y_2^2} \right) = 0 \quad \text{so bc's satisfied.}
\]

Note that \(G_0(y_1, x) = G_0(x, y)\) symmetric.

Ex: Work out \(G_N\) in some degree of detail:

Prob 9.5.16 of text: Find \(G\) for 60° wedge with 1 bc +

Find arrangement of image

by following reasoning,

i) Put usual image with \(-\) sign at \((y_1, y_2)\). We now satisfy bc along line \(\theta = 0\), but not at \(\theta = 60° = \pi/3\).

ii) Reflect the prev two images about line \(\theta = \pi/3\) with \(-\) sign. We get 4 sources. Satisfy bc at \(\theta = \pi/3\), but not at \(\theta = 0\).

iii) For this, use another reflection about \(\theta = 0\). We then have a total of 6 sources, and bc's are satisfied. The drawing gives a quite complete answer to the prob. We will not write down the sol in detail: \(G = G^2 - G^{(c)} + G^{(d)} - G^2 + G^2 - G^2\)